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YORKE AND WRIGHT 3/2-STABILITY THEOREMS FROM A UNIFIED POINT OF VIEW

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Abstract. We consider a family of scalar delay differential equations $x'(t) = f(t, x_t)$, with a nonlinearity f satisfying a negative feedback condition combined with a boundedness condition. We present a global stability criterion for this family, which in particular unifies the celebrated 3/2-conditions given for the Yorke and the Wright type equations. We illustrate our results with some applications.

1. **Introduction and main result.** In this paper we present a global stability criterion for a family of scalar functional differential equations

$$x'(t) = f(t, x_t), \quad (x_t(s) \stackrel{def}{=} x(t+s),\ s \in [-1, 0]), \tag{1}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is a measurable functional, $\mathcal{C} = \mathcal{C}([-1, 0], \mathbb{R})$.

As we will show in Section 3, our setting allows us to prove global stability results for a large family of functional delay differential equations, including a very general form of the delayed logistic equation, and differential equations with maxima among others.

Next we introduce the hypotheses that will be required in Eq. (1). In order to understand the motivation for the choice of these conditions, we recall some classical results. We refer to the famous 3/2 stability results due to Myshkis [9], Wright [13] and Yorke [14].

In particular, the Wright equation can be written in the form

$$x'(t) = f(x(t-1)), \tag{2}$$

with $f(x) = p(e^{-x} - 1)$, $p > 0$. The famous 3/2 stability result by Wright says that all solutions of this equation converge to zero if $p \le 3/2$. The closeness between this condition and the (local) asymptotic stability condition, $p < \pi/2$, suggests the

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equivalence between the local and the global asymptotic stability (this is the famous Wright's conjecture, which still remains open).

However, it is known that number 3/2 is the best bound when we consider differential equations with variable delay , even in the linear case

$$x'(t) = -px(t - h(t)) \,,\, p > 0,\, 0 < h(t) \leq h. \tag{3}$$

Myshkis proved that Eq. (3) is exponentially stable if $ph < 3/2$, and it is possible to find examples such that $ph = 3/2$ and (3) has a nontrivial periodic solution (see [14, p. 191]).

In 1970, Yorke extended the Myshkis criterion to a family of scalar functional differential equations (1) where $f : \mathbb{R} \times \mathcal{C}([-h, 0]) \to \mathbb{R}$, $h > 0$, is continuous and satisfies the following conditions:

(Y1) There exists $a < 0$ such that

$$a\mathcal{M}(\phi) \leq f(t, \phi) \leq -a\mathcal{M}(-\phi)$$

for all $\phi \in \mathcal{C}([-h, 0])$, where $\mathcal{M}(\phi) = \max\{0, \max_{s\in[-h,0]} \phi(s)\}$.

(Y2) For all sequences $t_n \to \infty$ and ϕ_n converging to a constant nonzero function in $\mathcal{C}([-h, 0])$, $f(t_n, \phi_n)$ does not converge to 0.

Under these conditions, if $0 < |a|h < 3/2$ then all solutions of (1) converge to zero as $t \to \infty$.

We notice that condition **(Y2)** is only required to guarantee that the solutions of (1) that monotonically converge to a constant in fact should converge to zero.

One can check that condition **(Y1)** is not satisfied by the Wright equation, and therefore Wright's theorem cannot be deduced from the Yorke result.

Trying to generalize the Wright theorem, in [7] we prove the following result:

Theorem 1. *Assume that $f \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ and it satisfies the following conditions:*

(A1) *$xf(x) < 0$ for $x \neq 0$ and $f'(0) < 0$.*

(A2) *f is bounded below and has at most one critical point $x^* \in \mathbb{R}$ which is a local extremum.*

(A3) *$(Sf)(x) < 0$ for all $x \neq x^*$, where $Sf = f'''(f')^{-1} - (3/2)(f'')^2(f')^{-2}$ is the Schwarz derivative of f.*

If $|f'(0)| \leq 3/2$, then the steady state solution $x(t) = 0$ of Eq. (2) is globally attracting.

Remark 1. Conditions **(A1)-(A3)** are satisfied by the Wright equation and other more complicated equations arising in population dynamics (see [7] for details).

Conditions **(A1)-(A2)** are not sufficient for the global attractivity in (2) (see [7, 12]). Hence an additional condition is required. We note that condition **(A3)** is not the unique option, in fact we only need some geometric consequences of the inequality $Sf < 0$ for the graph of f. In particular, the following key result ([7, Lemma 2.1]) is very important:

Lemma 1. *Assume that f satisfies* **(A1)-(A3)***, and $f''(0) > 0$. Let $a = f'(0) < 0$, $b = -f''(0)/(2f'(0)) > 0$, and $r(x) = ax/(1 + bx)$. Then $r(x) > f(x)$ if $x \in (-1/b, 0)$, and $r(x) < f(x)$ if $x > 0$.*

From Lemma 1, we can obtain immediately the following

Corollary 1. *Assume that f satisfies* **(A1)-(A3)***, and let $f(t, \phi) = f(\phi(-1))$. Then the following "generalized Yorke condition":*

(GY) *There exist $a < 0,\ b \geq 0$ such that*

$$r(\mathcal{M}(\phi)) = \frac{a\mathcal{M}(\phi)}{1 + b\mathcal{M}(\phi)} \leq f(t,\phi) \leq \frac{-a\mathcal{M}(-\phi)}{1 - b\mathcal{M}(-\phi)} = r(-\mathcal{M}(-\phi)),$$

where the first inequality holds for all $\phi \in \mathcal{C}$, and the second one for all ϕ such that $\min_{s\in[-1,0]} \phi(s) > -b^{-1} \in [-\infty, 0)$. Here $\mathcal{M}(\phi) = \max\{0, \max_{s\in[-1,0]} \phi(s)\}$ is the Yorke functional.

This corollary suggests the unification of Yorke's and Wright's 3/2 results by using condition **(GY)**. In fact, we introduce the following three hypotheses **(H)**:

(H1) $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}$ satisfies the Carathéodory condition. Moreover, for every $q \in \mathbb{R}$ there exists $\vartheta(q) \geq 0$ such that $f(t,\phi) \leq \vartheta(q)$ almost everywhere on $\mathbb{R}$ for every $\phi \in \mathcal{C}$ which satisfies the inequality $\phi(s) \geq q, s \in [-1,0]$.

(H2) Condition **(GY)** holds.

(H3) $\int_0^{+\infty} f(s,p_s)ds$ diverges for every continuous $p(s)$ having nonzero limit at infinity.

We recall that $f(t,\phi)$ is a Carathéodory function if it is measurable in t for each fixed ϕ, continuous in ϕ for each fixed t, and for any fixed $(t,\phi) \in \mathbb{R}\times\mathcal{C}$ there is a neighbourhood $V(t,\phi)$ and a Lebesgue integrable function m such that $|f(s,\psi)| \leq m(s)$ for all $(s,\psi) \in V(t,\phi)$ (see [3, p. 58]).

The following result improves the above mentioned theorems due to Wright and Yorke respectively (notice that **(H3)** implies that $x(t) \equiv 0$ is the unique equilibrium of the equation):

Theorem 2. *Assume that f satisfies* **(H)** *and either $b > 0$ and $|a| \leq 3/2$, or $b = 0$ and $|a| < 3/2$. Then all solutions of (1) converge to zero as $t \to +\infty$.*

Remark 2. 1. If **(H2)** holds with $b = 0$ (*Yorke condition*), then **(H1)** is satisfied automatically with $\vartheta(q) = -a\mathcal{M}(-q)$.

2. Conditions **(H1)**- **(H3)** are satisfied for Eq. (2) under our hypotheses **(A1)-(A3)**, with $a = f'(0)$, $b = -f''(0)/(2f'(0))$.
3. The constant 3/2 in Theorem 2 is the best possible. For $b = 0$, this sharpness was shown in [14] (see also [4, 8]). For $b > 0$, Theorem 2 can be applied to prove that all positive solutions of the logistic equation with variable delays

$$x'(t) = px(t)(1 - x(t - h(t)))\,,\ 0 < h(t) \leq h, \tag{4}$$

converge to 1 if $ph \leq 3/2$ (see Theorem 3 in Section 3). Since the linearized equation of (4) at $x = 1$ is (3), constant 3/2 cannot be improved. It is a remarkable fact that when Yorke's result cannot be extended to the value $a = -3/2$ for $b = 0$, Theorem 2 allows this for every $b > 0$.

If $b > 0$, since $\mathcal{M}$ is a positively homogeneous functional ($\mathcal{M}(k\phi) = k\mathcal{M}(\phi)$ for every $k \geq 0$, $\phi \in \mathcal{C}$), and since the global attractivity property of the trivial solution of (1) is preserved under the simple scaling $x = b^{-1}y$, the exact value of b does not have importance and we can assume that $b = 1$. Also, the change of variables $x = -y$ transforms (1) into $y'(t) = -f(t, -y_t)$ so that it suffices that at least one of the two functionals $f(t,\phi), -f(t,-\phi)$ satisfies **(GY)**.

2. **Proof of Theorem 2.**

2.1. **Auxiliary results.** Throughout this subsection, we will assume that $b = 1$ (and hence $r(x) = ax/(1+x)$).

Lemma 2. *Let* **(H)** *hold and* $x : [\alpha - 1, \omega) \to \mathbb{R}$ *be a solution of (1) defined on the maximal interval of existence. Then* $\omega = +\infty$ *and* $M = \limsup_{t\to\infty} x(t)$, $m = \liminf_{t\to\infty} x(t)$ *are finite. Moreover, if* $m \geq 0$ *or* $M \leq 0$, *then* $M = m = 0$.

Proof. Note that **(GY)** implies that $f(t, \phi) \geq a$ for all $t \in \mathbb{R}$ and $\phi \in \mathcal{C}$. We claim that every solution $x(t, \gamma)$ with initial value γ such that $q \leq \gamma(s) \leq Q$, $s \in [-1, 0]$, satisfies the inequality

$$x(t, \gamma) \geq \min\{q, 0\} + a = \kappa, \quad t \geq 0.$$

Indeed, if there is $\delta > 0$ such that $x(t, \gamma) = \min\{q, 0\} + (1 + \delta)a$ for the first time at some point $t = u \geq 0$, then, for every $w \in (u - 1 - \delta/2, u)$,

$$x(u, \gamma) - x(w, \gamma) = \int_w^u f(s, x_s(\gamma))ds \geq a(u - w) > a(1 + \delta/2).$$

Hence $x(w, \gamma) \leq a\delta/2 < 0$ for all $w \in (u - 1 - \delta/2, u)$, and therefore $\mathcal{M}(x_s) = 0$ for all $s \in (u - \delta/2, u)$. Thus, by **(H2)**, $x'(s, \gamma) = f(s, x_s(\gamma)) \geq 0$ within some left neighborhood of u, contradicting the choice of this point.

Proceeding analogously and using **(H1)**, we obtain that

$$x(t, \gamma) \leq \max\{Q, 0\} + \vartheta(\kappa), \quad t \geq 0.$$

Hence $x(t)$ is bounded on the maximal interval of existence that implies the boundedness of the right hand side of Eq. (1) along $x(t)$. Thus $\omega = +\infty$ due to the corresponding continuation theorem (see [3]).

Now, suppose that $M < 0$. Then $x(t) < M/2 < 0$ beginning from some $t' = d - 1$ so that, in view of $x'(t) \geq 0$, the solution $x(t)$ converges monotonically to the negative value $x(+\infty) = M$. We get a contradiction since, by **(H3)**,

$$x(t) = x(d) + \int_d^t f(s, x_s)ds \to +\infty.$$

Next we consider the situation when $M = 0$ and $m < 0$. In this case, $x(t)$ necessarily oscillates about zero. Indeed, otherwise $x(t) \leq 0$ and thus $x'(t) = f(t, x_t) \geq 0$, so that $x(t)$ converges monotonically to the trivial steady state (implying $m = 0$). Now, since $x(t)$ is oscillating, we can find a sequence of intervals $I_k = (l_k, r_k)$ containing e_k such that $x(t) < 0$, $t \in I_k$ and $\min_{I_k} x(t) = x(e_k) \to m$ as $k \to +\infty$, while e_k is the minimal point from I_k having this property. We claim that $e_k - l_k \leq 1$. On the contrary, let us suppose that $e_k - l_k > 1$. Then $x_t < 0$ (and, consequently, $x'(t) \geq 0$) for all t in a small neighborhood of e_k, contradicting to the choice of e_k as the leftmost point of global minimum in I_k. Finally, observing that

$$x(e_k) = \int_{l_k}^{e_k} f(s, x_s)ds \geq r(\max_{u\in[l_k-1,l_k]} x(u)) \to 0, \ k \to +\infty,$$

we get a contradiction with the relation $x(e_k) \to m < 0$.

The case $m \geq 0$ is similarly addressed. □

Remark 3. The last part of the above proof can be repeated to analyze the structure of the set of extreme points for every solution $x(t)$ satisfying $m < 0$ and $M > 0$. We see that in that case there exist sequences of intervals $A_k = (a_k, b_k)$, $A'_k = (a'_k, b'_k)$ and points $e_k \in A_k$, $e'_k \in A'_k$ such that $x(a'_k) = x(a_k) = 0$, $e_k - a_k \leq 1, e'_k - a'_k \leq, 1$ while $x(e_k) \to m$, $x(e'_k) \to M$ and $x(t)$ does not change sign over

A_k, A'_k. Moreover, for each k, e_k and e'_k could be chosen as the points of global maximum of $|x(t)|$ on A_k and A'_k respectively.

Now, we define continuous functions $A, B : (-1, +\infty) \to \mathbb{R}$ and $D : \mathbb{R}_+ \to \mathbb{R}$ by

$$A(x) = x + r(x) + \frac{1}{r(x)} \int_x^0 r(t)dt, \quad B(x) = \frac{1}{r(x)} \int_{-r(x)}^0 r(s)ds \quad \text{for } x \neq 0,$$

$$A(0) = B(0) = 0, \quad D(x) = \begin{cases} A(x) & \text{if } r(x) < -x, \\ B(x) & \text{if } r(x) \geq -x. \end{cases}$$

For the case $a < -1$, we will also use the rational function

$$R(x) = \frac{(A'(0))^2\ x}{A'(0) - (A''(0)/2)x}$$

defined on the interval $(2A'(0)/A''(0), \infty) = (\nu, \infty)$. Note that $A'(0) = a + 1/2 < 0$, $A''(0) = -(2a + 1/3) > 0$. It is easy to check that $a < -1/6$ implies $(\nu, +\infty) \subset (-1, +\infty)$, and that $A(x_2) = B(x_2)$, where $r(x_2) = -x_2 < 0$. Also $B'(0) = -(r'(0))^2/2 = -a^2/2$.

The following relations were established in [7, Lemma 2.4, Corollary 2.7]:

Lemma 3. *If $a < -1$, then $(A(x) - R(x))x > 0$ for $x \in (\nu, x_2) \setminus \{0\}$. Moreover, if $a \in [-1.5, -1.25]$, then $D(x) > R(x)$ for $x > 0$.*

In the next stage of the proof, we establish various relations between m and M, all of them being expressed in terms of the recently introduced functions. Notice that, by Lemma 2, the only case of interest is when $m < 0 < M$; thus we can suppose the existence of points t_j, s_j of local maxima and local minima respectively such that $x(t_j) = M_j \to M, x(s_j) = m_j \to m$ and $s_j, t_j \to +\infty$ as $j \to \infty$.

Lemma 4. *We have $m \geq D(M)$ and $m \geq r(-r(M)/2)$.*

Proof. First we assume that $r(M) < -M$. Then $M/r(M) \in (-1, 0]$. Take now $\varepsilon > 0$ such that $t_1 = (M + \varepsilon)/r(M + \varepsilon) \in (-1, 0]$. Obviously, $m_j > m - \varepsilon$ and $M_j < M + \varepsilon$ for sufficiently large j. By Remark 3, there exists $\tilde{s}_j \in [s_j - 1, s_j]$ such that $x(\tilde{s}_j) = 0$ and $x(t) < 0$ for $t \in (\tilde{s}_j, s_j]$.

Next, $z(t) = r(M + \varepsilon)(t - \tilde{s}_j),\ t \in [\tilde{s}_j + t_1, \tilde{s}_j + t_1 + 1]$ solves the initial value problem $z(s) = M + \varepsilon,\ s \in [\tilde{s}_j + t_1 - 1, \tilde{s}_j + t_1]$ for

$$z'(t) = r(z(t-1)). \tag{5}$$

Clearly $M + \varepsilon = z(t) > x(t)$ for all $t \in [t_1 + \tilde{s}_j - 1, \tilde{s}_j + t_1]$. Moreover, we will prove that $z(t) > x(t)$ for all $t \in [\tilde{s}_j + t_1, \tilde{s}_j)$.

Indeed, if this is not the case we can find $t_* \in [\tilde{s}_j + t_1, \tilde{s}_j)$ such that $z(t_*) = x(t_*)$ and $z(t) > x(t)$ for all $t \in [\tilde{s}_j + t_1, t_*]$. We claim that

$$x'(t) > z'(t) \ \text{ for all } \ t \in [t_*, \tilde{s}_j]. \tag{6}$$

We have

$$z'(t) = r(z(t-1)) < r(\mathcal{M}(x_t)) \leq f(t, x_t) = x'(t).$$

After integration over $(t_*, \tilde{s}_j)$, it follows from (6) that $x(t_*) < z(t_*)$, which is a contradiction.

Thus $z(t) > x(t)$ for $t \in [t_1 + \tilde{s}_j, \tilde{s}_j)$ and, arguing as above, we obtain

$$\begin{aligned} m_j &= \int_{\tilde{s}_j}^{s_j} x'(s)ds = \int_{\tilde{s}_j}^{s_j} f(s, x_s)ds \geq \int_{\tilde{s}_j}^{s_j} r(\mathcal{M}(x_s))ds > \int_{\tilde{s}_j}^{s_j} r(z(s-1))ds \\ &= \int_{\tilde{s}_j - 1}^{\tilde{s}_j + t_1} r(M+\varepsilon)ds + \int_{\tilde{s}_j + t_1}^{s_j - 1} r(r(M+\varepsilon)(s - \tilde{s}_j))ds \\ &= r(M+\varepsilon)(t_1 + 1) + \int_{t_1}^{s_j - \tilde{s}_j - 1} r(r(M+\varepsilon)u)du \\ &\geq M + \varepsilon + r(M+\varepsilon) + \int_{t_1}^{0} r(r(M+\varepsilon)u)du = A(M+\varepsilon). \end{aligned}$$

As a limit form of this inequality, we get $m \geq A(M)$.

In the general case (i.e. we do not assume that $r(M) < -M$), we use the inequality $f(t, x_t) \geq r(\mathcal{M}(x_t)) > r(M+\varepsilon)$ to see that, for $t \in (\tilde{s}_j - 1, \tilde{s}_j)$,

$$x(t) = -\int_t^{\tilde{s}_j} x'(s)ds < -\int_t^{\tilde{s}_j} r(M+\varepsilon)ds = r(M+\varepsilon)(t - \tilde{s}_j).$$

In consequence,

$$\begin{aligned} m_j &= x(s_j) = \int_{\tilde{s}_j}^{s_j} f(s, x_s)ds > \int_{\tilde{s}_j}^{s_j} r(r(M+\varepsilon)(s - \tilde{s}_j - 1))ds \\ &\geq \int_0^{s_j - \tilde{s}_j} r(r(M+\varepsilon)(s-1))ds \geq \int_0^1 r(r(M+\varepsilon)(u-1))du = B(M+\varepsilon). \end{aligned}$$

Therefore, we obtain that $m \geq B(M)$. Finally, applying Jensen's integral inequality (see [10, p. 110]) , we have

$$m \geq B(M) = \frac{1}{r(M)} \int_{-r(M)}^{0} r(s)ds \geq r(-r(M)/2).$$

This completes the proof. □

As a consequence of Lemmas 3, 4, we obtain that $R(m), r(m)$ and $r(r(-r(M)/2))$ are well-defined and that $R(\nu, +\infty) \subset (\nu, +\infty)$ for suitable values of a:

Corollary 2. *We have* $m > -1, r(-r(M)/2) > -1$ *if* $a \in [-1.5, 0)$ *and* $m > \nu$ *if* $a \in [-1.5, -1.25]$.

Proof. Indeed, for $a \in (-2, 0)$ we have

$$m \geq r(-r(M)/2) > r(-\frac{r(+\infty)}{2}) = \frac{-a^2}{2-a} \geq -1.$$

Next, $-A'(0) = -(a + 0.5) \leq 1$ for $a \geq -1.5$, and Lemmas 3, 4 lead to the estimate

$$m > D(+\infty) = B(+\infty) \geq R(+\infty) = -A'(0)\nu \geq \nu.$$

This proves the corollary. □

Lemma 5. *Let* $a \in [-1.5, 0)$. *We have* $M \leq r(m)$. *Moreover, if* $a \in [-1.5, -1.25]$ *then* $M < R(m)$.

Proof. We have that $r(m)$ is well defined and $[m, +\infty) \subset [-1, +\infty)$ since $a \in [-1.5, 0)$ (see Corollary 2). Let us consider a solution $x(t)$ of Eq. (1) and take s_j, t_j, m_j, M_j, as in the paragraph below Lemma 3. By Remark 3, there exists $\tilde{t}_j \in [t_j - 1, t_j)$ such that $x(\tilde{t}_j) = 0$.

First we note that, for $\varepsilon > 0$ and j sufficiently large, $f(s, x_s) \le r(-\mathcal{M}(-x_s)) < r(m-\varepsilon)$ for $s \in [\tilde{t}_j - 1, \tilde{t}_j]$ and $a \in [-1.5, 0)$. Thus

$$M_j = x(t_j) = \int_{\tilde{t}_j}^{t_j} f(s, x_s)ds < \int_0^1 r(m-\varepsilon)ds = r(m-\varepsilon).$$

By taking the limits as $\varepsilon \to 0$ and $j \to +\infty$, we obtain the inequality $M \le r(m)$.

Now, if $a < -1$ then $r(m) > -m$, from which for all sufficiently small $\varepsilon > 0$ we obtain that $t_2 = (m-\varepsilon)(r(m-\varepsilon))^{-1} \in (-1, 0]$. Next, $z(t) = r(m-\varepsilon)(t - \tilde{t}_j)$, with $t \in [\tilde{t}_j + t_2, \tilde{t}_j + t_2 + 1]$, solves the initial value problem $z(s) = m - \varepsilon$, $s \in [\tilde{t}_j + t_2 - 1, \tilde{t}_j + t_2]$ for Eq. (5). Now we only have to argue as in the proof of Lemma 4 to find out that

$$M_j = \int_{\tilde{t}_j}^{t_j} f(t, x_t)dt \le \int_{\tilde{t}_j}^{t_j} r(-\mathcal{M}(-x_t))dt \le \int_{\tilde{t}_j}^{t_j} r(z(t-1))dt \le A(m-\varepsilon).$$

Thus $M \le A(m)$. Finally, by Lemma 3 and Corollary 2, we obtain $M \le A(m) < R(m)$ when $a \in [-1.5, -1.25]$. □

2.2. **Proof of Theorem 2.** Let $x : [\alpha - h, +\infty) \to \mathbb{R}$ be a solution of Eq. (1) and set $M = \limsup_{t\to\infty} x(t)$, $m = \liminf_{t\to\infty} x(t)$. We will reach a contradiction if we assume that $m < 0 < M$ (note that the cases $M \le 0$ and $m \ge 0$ were already considered in Lemma 2).

Assume first that $b = 1$. If $a \in (-1.5, 0)$, in view of Lemmas 4, 5 and Corollary 2 we obtain that $M \le r(m) \le r \circ r(-r(M)/2) = \lambda(M)$ with the rational function $y = \lambda(x)$. Now, $\lambda(M) < M$ for $M > 0$ if $\lambda'(0) = (1/2)|a|^3 < 1$. Therefore, if $a \in [-1.25, 0)$ we obtain the desired contradiction under the assumption $M > 0$.

Now let $a \in [-1.5, -1.25]$ and, consequently, $R'(0) = a + 0.5 \in [-1, -0.75]$. In this case Lemmas 3, 4 and 5 imply that $M < R(R(M))$. As $R \circ R(x) \le x$ for all $x > 0$ whenever $(R \circ R)'(0) = (R'(0))^2 \le 1$, we obtain a contradiction again. Therefore $x(t) \equiv 0$ is the global attractor of Eq. (1) if $a \in [-1.5, 0)$.

If $b = 0$, we employ the linear function $r(x) = ax$. Arguing as in the proofs of Lemmas 4 and 5, we obtain that $m \ge (a + 1/2)M$, $M \le (a + 1/2)m$ for $a \in (-3/2, -1]$, and $m \ge (-1/2)a^2 M$, $M \le am$ for $a < 0$ (see [7] for more details).

Hence, if $a \in (-3/2, -1]$, we get the contradiction $M \le (a + 1/2)^2 M < M$. Finally, if $a \in (-1, 0)$, we obtain $m \ge (-1/2)a^2 M \ge (-1/2)a^3 m$. Thus, $-a^3/2 \ge 1$, a contradiction.

3. **An application.** Probably, the most interesting object to which we can apply our results is the following generalization of the logistic delayed equation:

$$x'(t) = \lambda(t)x(t)f(t, \mathcal{L}(t, x_t)), \qquad t \ge 0,\ x \ge 0. \tag{7}$$

Here $\lambda : [-h, \infty) \to (0, \infty)$ is measurable and $\int_0^\infty \lambda(s)ds = \infty, \sup_{t\ge 0} \int_{t-h}^t \lambda(s)ds < \infty$. We suppose that $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{L}(t, \phi) : \mathbb{R}_+ \times \mathcal{C}([-h, 0]) \to \mathbb{R}$ are Carathéodory functions, and that

$$\min \phi \le \mathcal{L}(t, \phi) \le \max \phi$$

for every $\phi \in \mathcal{C}([-h, 0])$, $t \ge 0$. We are interested in the case when, apart from $x(t) \equiv 0$, Eq. (7) has another equilibrium $x(t) \equiv \kappa$. Without loss of generality we can assume that $\kappa = 1$ (and, consequently, that $f(t, 1) \equiv 0$). Finally, we will require the divergence of $\int_0^{+\infty} \lambda(s)f(s, w(s))ds$ for every continuous function w converging to some positive number different from 1, as well as the following negative feedback condition:

$$x\ f(t, 1+x) < 0\,, \forall\, x > -1,\ x \neq 0.$$

As a simple observation shows, every nontrivial solution of (7) is eventually positive, so that we will only study the behavior of the positive solutions.

The next result is a consequence of Theorem 2:

Theorem 3. *Assume that*

$$x\ (f(t, x+1) - r(x)) \geq 0 \tag{8}$$

for some $r(x) = ax/(1+bx)$ *with* $a < 0,\ b \geq 0$, *and for all* $x > \max\{-1, -b^{-1}\}$.

If $b \neq 0.5$ *and, for some* $T > 0$, $|a|\Lambda \leq 3/2$ *holds with* $\Lambda = \sup_{t\geq T}\int_{t-h}^{t}\lambda(s)ds$, *then* $\lim_{t\to+\infty} x(t) = 1$ *for every nontrivial solution of (7). If* $b = 0.5$, *then the same conclusion holds when* $|a|\Lambda < 3/2$.

Proof. First, let $b \geq 0.5$. The change of variables $y(s) = \ln x(t)$, where $s = s(t) = \Lambda^{-1}\int_0^t \lambda(u)du$ (with the inverse $t = t(s)$ to $s = s(t)$), reduces (7) to

$$y'(s) = \Lambda g(s, \mathcal{K}(s, y_s)), \quad y \in \mathbb{R}, \quad s \geq 0. \tag{9}$$

Here, $g(s,x) = f(t(s), x)$ and $\mathcal{K}(s,\phi) = \mathcal{L}(t(s), \Psi)$, where $\Psi = \Psi(s,\phi)$ is defined by $[\Psi(s,\phi)](u) = \exp[\phi(-\sigma(s,u))]$, being $\sigma(s,u) = \Lambda^{-1}\int_{t(s)+u}^{t(s)}\lambda(v)dv$. Notice that $\mathcal{K} : \mathbb{R}_+ \times \mathcal{C}([-1,0]) \to \mathbb{R}$ is a Carathéodory function, $\exp(-\mathcal{M}(-\phi)) \leq \mathcal{K}(s,\phi) \leq \exp(\mathcal{M}(\phi))$ for every $\phi \in \mathcal{C}([-1,0])$, $s \geq 0$, and $\mathcal{K}(s,0) \equiv 1$. Since $g(s, \mathcal{K}(s,\phi)) = f[t(s), 1 + (\mathcal{K}(s,\phi) - 1)]$ we have, for $\mathcal{K}(s,\phi) \leq 1$, that

$$\begin{aligned} r_1(\mathcal{M}(\phi)) &\leq 0 \leq \Lambda g(s, \mathcal{K}(s,\phi)) \leq \Lambda r(\mathcal{K}(s,\phi) - 1) \\ &\leq \Lambda r(\exp[-\mathcal{M}(-\phi)] - 1) \leq r_1(-\mathcal{M}(-\phi)), \end{aligned} \tag{10}$$

where $r_1(x) = a\Lambda x/(1 + (b - 0.5)x)$. Indeed, function $v(x) = r(e^x - 1)$ satisfies conditions **(A1)-(A3)** in Theorem 1 and therefore, by Lemma 1, $r(e^x - 1) \leq r_1(x)$ for $x < 0$, and $r(e^x - 1) \geq r_1(x)$ for $x > 0$, since $v'(0) = a$, $v''(0) = a(1 - 2b)$.

Analogously, if $\mathcal{K}(s,\phi) \geq 1$ then

$$\begin{aligned} r_1(-\mathcal{M}(-\phi)) &\geq 0 \geq \Lambda g(s, \mathcal{K}(s,\phi)) \geq \Lambda r(\mathcal{K}(s,\phi) - 1) \\ &\geq \Lambda r(\exp[\mathcal{M}(\phi)] - 1) \geq r_1(\mathcal{M}(\phi)). \end{aligned} \tag{11}$$

Let now $b \in [0, 0.5]$. In this case, the change of variables $z(s) = -\ln x(t)$, where $s = s(t)$ reduces (7) to the form

$$z'(s) = -\Lambda g(s, \mathcal{K}(s, -z_s)), \quad z \in \mathbb{R}, \quad s \geq 0. \tag{12}$$

We have, for $\mathcal{K}(s,-\phi) \geq 1$, that

$$\begin{aligned} r_2(\mathcal{M}(\phi)) &\leq 0 \leq -\Lambda g(s, \mathcal{K}(s,-\phi)) \leq -\Lambda r(\mathcal{K}(s,-\phi) - 1) \\ &\leq -\Lambda r(\exp[\mathcal{M}(-\phi)] - 1) \leq r_2(-\mathcal{M}(-\phi)), \end{aligned} \tag{13}$$

where $r_2(x) = a\Lambda x/(1 + (0.5 - b)x)$. (Here we use $w(x) = -r(e^{-x} - 1)$ and argue as before). Next, if $\mathcal{K}(s,-\phi) \leq 1$, we obtain

$$\begin{aligned} r_2(-\mathcal{M}(-\phi)) &\geq 0 \geq -\Lambda g(s, \mathcal{K}(s,-\phi)) \geq -\Lambda r(\mathcal{K}(s,-\phi) - 1) \\ &\geq -\Lambda r(\exp[-\mathcal{M}(\phi)] - 1) \geq r_2(\mathcal{M}(\phi)). \end{aligned} \tag{14}$$

Hence, both equations (9) and (12) are of Carathéodory type and satisfy **(GY)**. Now, if $\lim_{s\to+\infty} w(s) = w_* \neq 0$, then $\int_{\mathbb{R}_+} g(s, w(s))ds = \int_{\mathbb{R}_+}\lambda(t)f(t, w_1(t))dt$ for some $w_1(t)$ with $\lim_{t\to+\infty} w_1(t) = \exp(w_*) \neq 1$ so that **(H3)** is also fulfilled.

Finally, the boundedness requirement from **(H1)** follows easily from (10), (11), (13), (14). In this way, we can finish the proof of Theorem 3 by applying Theorem 2 to Eqns. (9) and (12). $\square$

Remark 4. Notice that Theorem 3 still remains true if, in its statement, we replace the rational function $r(x)$ by any decreasing function $\tilde{r} : (-1, \infty) \to \mathbb{R}$ satisfying (8), and such that $\tilde{r}'(0) = a < 0$ and $\tilde{r}$ is below bounded if $-\tilde{r}''(0)/(2\tilde{r}'(0)) \geq 1/2$. Additionally, we should suppose the negativity of $S\rho,\ \rho(x) = \tilde{r}(\exp(\pm x) - 1))$, in order to use Lemma 1 while evaluating (10), (11), (13), (14). The sign of the second derivative of $\tilde{r}$ (associated with the sign of b before) does not matter now since $\tilde{r}(x)$ is defined for all $x > -1$. It is clear that $|a|\Lambda \leq 3/2$ implies the global stability when $\rho''(0) \neq 0$, while for $\rho''(0) = 0$ we should assume that $|a|\Lambda < 3/2$.

Example 1. Let us apply Theorem 3 to study the food-limited population model [1, 2, 5, 6, 11] with variable, continuous and generally unbounded delay $h(t) \geq 0$ such that $\inf_{\mathbb{R}_+}(t - h(t)) = h_*$ is finite (obviously, $h_* \leq 0$):

$$N'(t) = \lambda(t)N(t)\frac{k - N^l(t - h(t))}{k + \nu(t)N^l(t - h(t))},\ t \geq 0\ . \tag{15}$$

Here $k, l > 0$, $\lambda \in \mathcal{C}([h_*, \infty), (0, \infty))$, $\nu \in \mathcal{C}([0, \infty), [0, \infty))$ and it is assumed that

$$\int_0^\infty \frac{\lambda(s)}{1 + \nu(s)} ds = \infty. \tag{16}$$

Corollary 3. *Let $\nu_0 = \inf_t \nu(t) \geq 0$, and assume that for some $T \geq 0$ either $\nu_0 \neq 1$ and $l\Lambda/(1 + \nu_0) \leq \frac{3}{2}$, or $\nu_0 = 1$ and $l\Lambda/(1 + \nu_0) < \frac{3}{2}$, where $\Lambda = \sup_{t \geq T}\left\{\int_{t-h(t)}^t \lambda(s)ds\right\}$. Then every positive solution of equation (15) converges to $k^{1/l}$.*

Proof. With $N(t) = k^{1/l}\ x(s)$ and $s = s(t) = \Lambda^{-1}\int_0^t \lambda(u)du$, Eq. (15) is transformed into $x'(s) = \Lambda x(s) f(s, \mathcal{L}(s, x_s))$, where $\mathcal{L}(s, \phi) = \phi(-\Lambda^{-1}\int_{t(s)-h(t(s))}^{t(s)} \lambda(u)du)$, and $f(s, x) = (1 - x^l)/(1 + \nu(t(s))x^l)$.

In order to apply Theorem 3 to the *transformed* equation, we consider function

$$f^*(s, x) = f(t(s), x + 1) = \frac{1 - (x + 1)^l}{1 + \nu(t(s))(x + 1)^l}, \quad x > -1.$$

It is obvious that $xf^*(s, x) < 0$ for all $x \neq 0$. Next, $(1 - (x + 1)^l)/(1 + u(x + 1)^l)$ is increasing with respect to u for $x > 0$ fixed and decreasing with respect to u for fixed $x < 0$. Hence, $f^*(s, x) \geq \tilde{r}(x)$ for $x > 0$, and $f^*(s, x) \leq \tilde{r}(x)$ for $x < 0$, where

$$\tilde{r}(x) = \frac{1 - (x + 1)^l}{1 + \nu_0(x + 1)^l}, \quad x > -1.$$

Finally, it is easy to check that $\tilde{r}$ and ρ satisfy all conditions indicated in Remark 4. Notice that $a = \tilde{r}'(0) = -l/(1 + \nu_0) < 0$ and $\rho''(0) = 0$ only if $\nu_0 = 1$. $\square$

Remark 5. In the particular case when the delay is constant, Corollary 3 was proved in [2] under conditions (16) and $l\Lambda/(1 + \nu_0) \leq 1$. Also, Liu in [6] considered

$$N'(t) = \lambda(t)N(t)\left(\frac{k - N(t - h)}{k + \nu(t)N(t - h)}\right)^\beta,\ t \geq 0, \tag{17}$$

where $\beta = (2m + 1)/(2n + 1) \geq 1$, $m, n \in \mathbb{N}$, getting our 3/2 condition only in the particular case when $\nu_0 = 1$ (see [6, Theorem 2]). We remark that, for $\beta > 1$, the global attractivity of (17) can be always proved once (16) is assumed. Notice also that the statement of Corollary 3 remains true if we replace all entries of $N^l(t-h(t))$ in (15) by $N^l(\mu t)$, with $\mu \in (0,1)$, or by $(\max_{u\in[t-h_0,t]} N(u))^l$ for some $h_0 > 0$ (thus we can include in our considerations equations with linearly transformed argument and equations with maxima).

Remark 6. Assume (16). In [11], it has been established that the steady state of (15) with $h(t) = h$ is (locally) uniformly and asymptotically stable if

$$l\int_{t-h}^{t} \frac{\lambda(s)}{1+\nu(s)} ds \le \alpha < 3/2\,,\ t \ge h. \tag{18}$$

This inequality is less restrictive than the one we obtained in Corollary 3; thus, inspired by the Wright conjecture about the equivalence of global and local asymptotic stability, one can try to improve our result up to (18). However, it would be impossible: even with (18) satisfied, Eq. (15) can have nontrivial periodic solutions.

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