Global Stability of a Class of Scalar Nonlinear Delay Differential Equations

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Abstract

The problem of global stability in scalar delay differential equations of the form

$$x'(t) = f_1(x(t-h))g_2(x(t)) - f_2(x(t-h))g_1(x(t))$$

is studied. Functions f_i and $g_i, i = 1, 2$, are continuous and such that the equation assumes a unique positive equilibrium. Two types of sufficient conditions for the global asymptotic stability of the unique equilibrium are established: (i) delay independent, and (ii) conditions involving the size h of the delay. Delay independent conditions make use of the global stability in the limiting (as $h \to \infty$) difference equation $f_1(x_n)g_2(x_{n+1}) = f_2(x_n)g_1(x_{n+1})$: the latter always implying the global stability in the differential equation for all values of the delay $h \geq 0$. The delay dependent conditions involve the global attractivity property in specially constructed one-dimensional maps (difference equations) that include the nonlinearities f_i and g_i, and the delay h.

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1 Introduction

In this paper we study the global stability properties of the nonlinear differential delay equations

$$x'(t) = \begin{vmatrix} f_1(x(t-h)) & g_1(x(t)) \\ f_2(x(t-h)) & g_2(x(t)) \end{vmatrix}, \; x \in [0, +\infty) \overset{def}{=} \mathbb{R}_+. \tag{1.1}$$

Here $f_i, g_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and some or all of the following hypotheses are assumed throughout the paper as appropriate:

(H0) $g_1(0) = 0$, and $f_i(x) > 0, g_i(x) > 0$ for all $x > 0, i = 1, 2$;

(H1) $g(x) = g_1(x)/g_2(x)$ is strictly increasing for $x > 0$.
Furthermore $g(0+) = 0$, and $\lim_{x\to+\infty} g(x) = +\infty$;

(H2) Let $f(x) = f_1(x)/f_2(x)$. There is exactly one point $\overline{x} > 0$ such that $f(\overline{x}) = g(\overline{x})$; moreover, $f(x) > g(x)$ in $(0, \overline{x})$ and $f(x) < g(x)$ in $(\overline{x}, \infty)$.

Define $F(x) := g^{-1}(f(x))$. We also allow for the possibility that $\limsup_{x\to 0+} f(x) = +\infty$ (equivalently $\limsup_{x\to 0+} F(x) = +\infty$), however, in this case we will require that $\inf_{x\geq 0} F(x) > 0$.

The family of equations (1.1) includes such important particular cases as

$$x'(t) = -x(t) + f(x(t-h)) \tag{1.2}$$

and

$$x'(t) = f(x(t-h)) \quad \text{or} \quad x'(t) = x(t)f(x(t-h)). \tag{1.3}$$

A great deal of work is done on studying the global stability and oscillation properties of Eq. (1.2), let us mention here only a few of relevant references: [2, 3, 4, 5, 6, 7, 12, 16, 17, 19]. Starting with [19], Eq. (1.2) sometimes is considered under the assumption of negative Schwarz derivative $(Sf)(x)$ of the nonlinearity $f = f_1$, where

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

See also the papers [3, 6, 14, 16, 17]. To the best of our knowledge, the following result for Eq. (1.2) with f satisfying the negative Schwarzian assumption provides the best global stability conditions (at least for the monotone nonlinearity f):

Proposition 1 ([16]) *Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly decreasing, bounded, and $Sf(x) < 0,\ x \in \mathbb{R}_+$. Then the unique equilibrium $x(t) \equiv \overline{x}$ of Eq. (1.2) is globally attracting if $-\exp(-h)/f'(\overline{x}) > \ln\left((f'(\overline{x})^2 - f'(\overline{x}))(1 + f'(\overline{x})^2)^{-1}\right)$.*

An analog of this result for Eq. (1.3) is known as the 3/2-stability condition. It can be found in [15], where several concrete applications are also studied. If the nonlinearity f in (1.2) is not necessarily monotone, we can also use the following stability condition established in [6]:

Proposition 2 ([6]) *Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is either i) strictly decreasing and bounded or ii) $f(0) = 0$ and f has only one critical point $x_* : f'(x_*) = 0$, which is exactly its global maximum. If $Sf(x) < 0,\ x \in \mathbb{R}_+ \setminus \{x_*\}$, then the unique positive equilibrium $x(t) \equiv \overline{x}$ of Eq. (1.2) is globally attracting if $|(1 - \exp(-h))f'(\overline{x})| < 1$.*

The purpose of this note is to establish, assuming if necessary the negative Sf, some analogs of the above results for Eq. (1.1). This goal can be achieved relatively easy while we study the absolute stability of the equation; in fact, our results here are optimal. On the other hand, it is more difficult to derive analogs of the above propositions. Here, we state only some rather general results which could be considered as an attempt to generalize Proposition 2 for the case of Eq. (1.1). However, in a particular case, we are able to get the following stability condition.

Theorem 1 *Suppose that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, $S\varphi(x) < 0$ for all $x \in \mathbb{R}_+$, and either $\varphi(0) > 0$ or $\varphi(0) = 0$ and φ is bounded. Then the unique positive equilibrium $x(t) \equiv \overline{x}$ of the differential delay equation*

$$x'(t) = 1 - x(t)\varphi(x(t-h)) \tag{1.4}$$

is globally attracting if $\quad \overline{x}^2(1 - \exp(-h/\overline{x}))\varphi'(\overline{x}) < 1.$

Notice that Eq. (1.4) can be transformed, via a simple change of variables, to the form $y'(t) = y(t)(-y(t) + \varphi(y(t-h)))$, which is even more resemblant of (1.2); however, Propositions 1 and 2 cannot be applied to this equation. Finally, note that our consideration of Eqs. (1.1), (1.4) and of

$$x'(t) = -g(x(t)) + f(x(t-h)) \tag{1.5}$$

as well as the above hypotheses (H0)-(H2) are motivated by specific applications (see the last section of this paper, and also [1, 3, 4, 6, 11, 12, 18] with further references therein).

The paper is organized as follows. Section 2 deals with the delay independent conditions for global asymptotic stability. It also contains a result on uniform persistence and boundedness of solutions under general assumptions. Section 3 deals with delay dependent conditions for global asymptotic stability. Theorem 1 follows from a general result for equation (1.1). As an application of Theorem 1 a well known model from respiratory dynamics is considered.

2 Delay independent conditions for global stability

2.1 Existence of solutions, positive invariance, persistence and boundedness

We start with an easy observation that under the assumption (H0) and the only additional requirement of $g(0+) = 0$ every solution $x(t)$ of the initial value problem $x(s) = \phi(s), s \in [-h, 0]$ with strictly positive $\phi \in C[-h, 0]$ is also strictly positive. Indeed, assume $x(\tau) = 0$ while $x(t) > 0, t \in [0, \tau)$ for some $\tau > 0$. Then there exists a sequence $\tau_n \uparrow \tau$ such that $0 > x'(\tau_n) = f_1(x(\tau_n - h))g_2(x(\tau_n)) - f_2(x(\tau_n - h))g_1(x(\tau_n))$. This implies that $f(x(\tau_n - h)) - g(x(\tau_n)) < 0$. After taking the limit as $n \to \infty$ one gets the contradiction: $0 < f(x(t - \tau)) \leq g(x(\tau)) = 0$. It can also be shown that an initial function with a finite number of zeros results in a solution that is eventually strictly positive. In the sequel we consider only the solutions that are strictly positive on the initial interval.

We are assuming that the nonlinearities $f_i, g_i, i = 1, 2$, are such that for every initial function $\phi \in C([-h, 0], \mathbb{R}_+)$ with $\phi(s) > 0\ \forall s \in [-h, 0]$ the corresponding solution $x(t) = x(t, \phi)$ of equation (1.1) exists for all $t \geq 0$.

Note that under the hypotheses (H0)-(H2) and the additional requirement that $g_i(x), i = 1, 2$, are Lipschitz continuous the solutions do exist for all $t \geq 0$ and are unique. This follows from the fact that for arbitrary initial function $\phi \in C([-h, 0], \mathbb{R}_+)$ one solves the initial value problem for the ordinary differential equation

$$x'(t) = f_1(\phi(t-h))g_2(x(t)) - f_2(x(\phi(t-h))g_1(x(t)), \quad x(0) = \phi(0), \quad t \in [0, h].$$

The local solution exists on some interval $[0, T]$ where $T > 0$ is determined by a common upper bound of the Lipschitz constants for g_1 and g_2. As it will be shown later in the paper, the condition $g(x) > f(x)$ for all large x (which is a part of assumption (H2)), implies that the solution $x(t, \phi)$ stays bounded from above for all $t \geq 0$ with the bound determined by $f_i, g_i, i = 1, 2$, and the initial function ϕ. Therefore, the uniform upper bound for the Lipschitz constants for g_1, g_2 exists along the solution, and the value of T is uniformly bounded away from zero (for an arbitrary but fixed initial function). The solution $x(t, \phi)$ then exists for all $t \geq 0$ by the method of step by step continuation.

Equation (1.1) is equivalent, via the change of variables $t = h \cdot s$, to the equation

$$\begin{aligned} \mu x'(t) &= \begin{vmatrix} f_1(x(t-1)) & g_1(x(t)) \\ f_2(x(t-1)) & g_2(x(t)) \end{vmatrix} \\ &= f_1(x(t-1))g_2(x(t)) - f_2(x(t-1))g_1(x(t)), \end{aligned} \tag{2.1}$$

where $\mu = 1/h$. The limiting case $\mu = 0$ ($h = +\infty$) in (2.1) results in difference equation

$$f(x_n) = g(x_{n+1}), \quad n \in \mathbb{Z}_+ \tag{2.2}$$

which can be solved explicitly for x_{n+1}:

$$x_{n+1} = g^{-1}(f(x_n)) := F(x_n), \quad n \in \mathbb{Z}_+. \tag{2.3}$$

Some dynamical properties of the one-dimensional map F can be translated to those of equation (2.1), for arbitrary positive values of the parameter μ. This is true, in particular, with regard to the *invariance* property, the *contraction* property, and the *global stability* property.

Let $I = [\alpha, \beta] \subset \mathbb{R}_+$ be a closed invariant under F interval, that is $F(I) \subseteq I$. Set $X := C([-1, 0], \mathbb{R}_+)$, and $X_I := \{\phi \in X : \phi(s) \in I\ \ \forall s \in [-1, 0]\}$.

Lemma 1 *(Invariance) Assume (H0)-(H1). The set X_I is invariant under equation (2.1). That is, for arbitrary $\phi \in X_I$ the corresponding solution $x(t) = x(t; \phi)$ satisfies $x(t) \in I$ for all $t \geq 0$ and every $\mu \geq 0$.*

Proof. Assume that $\tau \geq 0$ is the first point of exit of solution $x(t)$ from the interval $[\alpha, \beta]$ at the endpoint β. That is, $x(\tau) = \beta, \alpha \leq x(t) \leq \beta$ for $t < \tau$, and there exists a sequence $\tau_n \downarrow \tau$ such that $x'(\tau_n) > 0$ and $x(\tau_n) > \beta$. Equation (2.1) shows

$$0 < x'(\tau_n) = f_1(x(\tau_n - 1))g_2(x(\tau_n)) - f_2(x(\tau_n - 1))g_1(x(\tau_n)),$$

which is equivalent to

$$f(x(\tau_n - 1)) > g(x(\tau_n)) \quad \text{or} \quad F(x(\tau_n - 1)) > x(\tau_n) > \beta.$$

The latter is a contradiction to interval I being invariant under F. The case when τ is the first point of exit from I at the level α is treated similarly.

Lemma 2 *(Contraction Property) Assume (H0)-(H1). Let $J := [\gamma, \delta]$ be a closed interval such that $F(J) := J_1 := [\gamma_1, \delta_1] \subseteq J$. Let $\phi \in X_J$ be arbitrary and $x(t) = x(t, \phi), t \geq 0$, be the corresponding solution of equation (2.1).*

(i) If neither of the endpoints γ_1 or δ_1 is a fixed point of the map F, then there exists a finite $T = T(\phi) \geq 0$ such that $x(t) \in J_1$ for all $t \geq T$;

(ii) If δ_1 is a fixed point of the map F and γ_1 is not, then for any $\varepsilon > 0$ there exists a finite $T = T(\phi, \varepsilon) \geq 0$ such that $x(t) \in [\gamma_1, \delta_1 + \varepsilon]$ for all $t \geq T$. Likewise, if γ_1 is a fixed point of the map F and δ_1 is not, then $x(t) \in [\gamma_1 - \varepsilon, \delta_1]$ for all $t \geq T$;

(iii) If both γ_1 and δ_1 are fixed points of F then for any $\varepsilon > 0$ there exists a finite $T = T(\phi, \varepsilon)$ such that $x(t) \in [\gamma_1 - \varepsilon, \delta_1 + \varepsilon]$ for all $t \geq T$;

Proof. Note first that in view of the invariance property (Lemma 1) $x(t) \in J\, \forall t \geq 0$.

Assume first that $x(0) = \phi(0) \in J_1$. Then $x(t) \in J_1$ for all $t \geq 0$. This is proved exactly the same way as the invariance property, Lemma 2.1. We leave the details to the reader.

Assume next that $x(0) = \phi(0) \notin J_1$. To be definite, let $x(0) > \sup\{J_1\} = \delta_1$. Equation (2.1) then implies that $x'(0) < 0$, therefore $x(t)$ is decreasing in some right neighborhood of $t = 0$.

(i) In this case, there exists a finite $T = T(\phi) > 0$ such that $x(T) = \delta_1$, and one can argue then as in the case $x(0) \in J_1$. Note also that $x(t)$ is decreasing in $[0, T]$. Indeed, if not then $x(t) > \delta_1$ for all $t \geq 0$. Since $g(x) > \sup\{f(u), u \in J\}$ for all $x \geq \delta_1$, equation (2.1) shows that $x'(t) = f_1(x(t-1))g_2(x(t)) - f_2(x(t-1))g_1(x(t)) \leq 0$, therefore $x(t)$ is decreasing for all $t \geq 0$.

Let $\lim_{t\to\infty} x(t) = \delta_0 \geq \delta_1$. Since δ_1 is not a fixed point of the map F one has $\sup\{F(x), x \in [\delta_0, x(0)]\} < \delta_1$. Therefore,

$$x'(t) = \frac{f(x(t-1)) - g(x(t))}{f_2(x(t-1))g_2(x(t))} \leq -\varepsilon_1 < 0$$

for some $\varepsilon_1 > 0$ and all sufficiently large t. This is a contradiction to $x(t)$ having a finite limit.

The reasoning for the other possibility, $x(0) = \phi(0) < \inf\{J_1\} = \gamma_1$, is very similar to the one above and is omitted.

(*ii*) Exactly as in the case (*i*) above, it can be shown that for any $\varepsilon > 0$ satisfying $\delta_1 + \varepsilon < \delta$ there exists a finite $T = T(\phi, \varepsilon)$ such that $x(T) = \delta_1 + \varepsilon$ and $x(t)$ is decreasing in $[0, T]$. The interval $[\gamma, \delta_1 + \varepsilon]$ is invariant under F, therefore $x(t) \in [\gamma, \delta_1 + \varepsilon]\ \forall t \geq T$.

(*iii*) This part is proved by combining the two subcases of the case (*ii*).

The contraction property justifies the assumption (H2), that we will be making later in the paper, as Propositions 3 and 4 below show.

Proposition 3 *Let (H0)-(H1) be satisfied. Assume additionally* $f(x) < g(x)$ *for all* $x \in (0, \overline{x})$*, where* $f(\overline{x}) = g(\overline{x})$*. Then for every initial function* $\phi \in C([-1, 0], \mathbb{R}_+)$ *with* $0 < \phi(s) < \overline{x}\ \forall s \in [-1, 0]$ *the corresponding solution* $x(t)$ *has the property* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $M = \sup_{[-1,0]} \phi(s)$. Set $\gamma = 0, \delta = M < \overline{x}$.

The assumption $f(x) < g(x) \forall x \in (0, \overline{x})$ implies that for any $0 < M < \overline{x}$ the corresponding interval $J = [0, M]$ is mapped into itself under F. Moreover, $F(J) \supset J$, $\sup\{F(J)\} := M_1 < M$, and $\cap_{n\geq 0} F^n(J) = 0$. Therefore, by Lemma 2 (ii), there exists $T_1 \geq 0$ such that $x(t) \in F(J)\ \forall t \geq T_1$. The repeated application of this shows that there exists an increasing sequence $T_n \uparrow \infty$ such that $x(t) \in F^n(J)\ \forall t \geq T_n$. This obviously implies $\lim_{t\to\infty} x(t) = 0$.

Proposition 4 *Let (H0)-(H1) be satisfied. Assume additionally* $f(x) > g(x)$ *for all* $x \in (\overline{x}, \infty)$*, where* $f(\overline{x}) = g(\overline{x})$*. Then for every initial function* $\phi \in C([-1, 0], \mathbb{R}_+)$ *with* $\phi(s) > \overline{x}\ \forall s \in [-1, 0]$ *the corresponding solution* $x(t)$ *has the property* $\lim_{t\to\infty} x(t) = \infty$.

Proof. Let $m = \inf_{[-1,0]} \phi(s)$. Set $\gamma = m, \delta = \infty$, and $J_1 = [m, \infty)$. Then, since $f(x) > g(x), x \in (\overline{x}, \infty)$, one has that $F(J_1) \subset J_1$, and $\inf\{F(J_1)\} > m$. Moreover, $\lim_{n\to\infty} \inf\{F^n(J_1)\} = \infty$. Exactly as in the proof of the contraction property, one shows that given $\phi \in C([-1, 0], \mathbb{R}_+)$ with $\phi(s) > \overline{x}\ \forall s \in [-1, 0]$ there exists $T_1 = T_1(\phi) \geq 0$ such that $x(t) \in F(J_1)\ \forall t \geq T_1$. By induction, there exists a sequence $T_n \uparrow \infty$, such that $x(t) \in F^n(J_1)\ \forall t \geq T_n$. Therefore, $\lim_{t\to\infty} x(t) = \infty$.

Our next result shows that solutions of equation (2.1) are eventually uniformly bounded away from zero and the infinity.

Theorem 2 *(Persistence and Boundedness) Let (H0)-(H1) be satisfied. Assume additionally that* $f(x) > g(x)$ *for all* $x \in (0, x_0)$ *and* $f(x) < g(x)$ *for all* $x > x^0$ *for some* $x_0 > 0$ *and* $x^0 > 0$*. Then there is an interval* $I_* := [\alpha_*, \beta_*], \alpha_* > 0, \beta_* < \infty$*, determined by* $f(x)$ *and* $g(x)$*, with the following property: for every initial function* $\phi(s) \in C([-1, 0], \mathbb{R}_+)$ *with* $\phi(s) > 0\ \forall s \in [-1, 0]$ *there exists a finite time* $T = T(\phi)$ *such that the corresponding solution of equation (2.1) satisfies* $x(t) \in I_*$ *for all* $t \geq T$.

Proof. The proof is based on the existence of a globally attracting invariant interval of the map F and the application of the contraction property, Lemma 2.

We claim first that the assumptions of the theorem imply the existence of an invariant interval I of the map F which is also globally attracting. That is, $F(I) \subseteq I$ and for every point $x \in \mathbb{R}_+ \setminus \{0\}$ there exists a positive integer $n = n(x)$ such that $F^n(x) \in I$.

Indeed, the assumption $f(x) > g(x)\ \forall x \in (0, x_0)$ and $f(x) < g(x)\ \forall x \in (x^0, \infty)$ implies that the map F has at least one fixed point in $\mathbb{R}_+ \setminus \{0\}$. Let x_+ and x^+ be its smallest and largest fixed points, respectively. In the case $f(0) < \infty$ let $\sup\{F(x), x \in [0, x^0]\} := L \geq x^+$. If $f(0) > 0$ set $l := \inf\{F(x), x \in [0, L]\} > 0$ and define $I := [l, L]$. One also has $l \leq x_+$. If $f(0) = 0$ then either $f(x) < g(x_+)\ \forall x \in (0, x_+)$ or $\sup\{f(x), x \in [0, x_+]\} > g(x_+)$. In the former case set $l := \inf\{F(x), x \in [x_+, L]\} > 0$ and define $I := [l, L]$. In the latter case, there exists the leftmost point $\hat{x} > 0$ such that $f(\hat{x}) = g(x_+)$ and $f(x) < g(x_+)$ for all $x \in (0, \hat{x})$. Set $l := \inf\{F(x), x \in [\hat{x}, L]\} > 0$ and define $I := [l, L]$.

Interval I is invariant under F by the construction, i.e. $F(x) \in I\ \forall x \in I$. It is also a globally attracting interval of the map F in $\mathbb{R}_+ \setminus \{0\}$. Indeed, if $x < l$ and $f(0) > 0$ then $F(x) \in [l, L] = I$. If $f(0) = 0$ then $x = 0$ is a repelling fixed point. Therefore, for any $x \in (0, l)$ there exists a positive integer $N = N(x)$ such that $F^N(x) \in [l, L]$ and $F^n(x) < l\ \forall\ 0 < n < N$. If $x > \sup\{I\} = L$ then $F(x) < x$. Consider the trajectory of point x: $o(x) := \{F^n(x), n \geq 0\}$. If $\inf\{o(x)\} > \sup\{I\}$ then $F^n(x)$ is decreasing and $\lim_{n\to\infty}[F^n(x)] \in [x^0, \infty)$ is a fixed point of the map F, a contradiction. If $l \leq \inf\{o(x)\} \leq L$ then there exists a positive integer $N = N(x)$ such that $F^N(x) \in [l, L]$ and therefore, $F^n(x) \in I\ \forall\ n \geq N$. If $\inf\{o(x)\} < l$ then there exists a positive integer $N = N(x)$ such that $F^N(x) > L\ \forall 0 < n < N$ and $F^N(x) < l$. Then set $y := F^N(x)$ and repeat the above argument for the case $x < l$.

In the case $f(0+) = +\infty$ one has $\inf_{x\geq 0} F(x) := l > 0$, by the assumption made below the hypothesis (H2). Set $L := \sup\{F(x), x \in [l, x^0]\}$ and define $I := [l, L]$. As above, interval I is invariant by the construction and also is globally attracting under F. The latter is shown in the same way as the above consideration in the case $f(0) < \infty$. We leave the details to the reader.

Given I as defined above, choose $\varepsilon > 0$ such that $0 < \alpha_* := \inf\{I\} - \varepsilon < \beta_* := \sup\{I\} + \varepsilon$ and set $I_* := [\alpha_*, \beta_*]$. Interval I being globally attracting implies that for every closed interval $J \supset I$, $\cap_{n\geq 0} F^n(J) \subseteq I$. Therefore, there exists a finite positive integer $N = N(J)$ such that $F^N(J) \subset I_*$ and $\alpha_* < \inf\{F^N(J)\} \leq \sup\{F^N(J)\} < \beta_*$.

Finally, let $\phi(s) \in C([-1, 0], \mathbb{R}_+)$ with $\phi(s) > 0\ \forall s \in [-1, 0]$ be given. Set $m := \inf_{[-1,0]} \phi(s), M := \sup_{[-1,0]} \phi(s)$. Choose J to be a closed invariant interval such that $[m, M] \subset J$ and $I \subset J$. Then, by repeated application of the contraction property, Lemma 2, one shows that there exists a finite $T = T(N, J)$ such that $x(t) \in I_*\ \forall t \geq T$.

2.2 Global stability: sufficient conditions

The most important implication of the contraction property is the following global stability result for equation (2.1).

Theorem 3 *(Global Stability) Assume (H0)-(H1). Let $\overline{x}$ be an attracting fixed point of the map F with the domain J_0 of immediate attraction: $\lim_{n\to\infty} F^n(x) = \overline{x}$, $\forall x \in J_0$. Then, for every initial function $\phi(s) \in X_{J_0} = C([-1, 0], J_0)$ the corresponding solution $x(t) = x(t, \phi)$ has the property $\lim_{t\to\infty} x(t) = \overline{x}$.*

Proof. Note that J_0 is open in $\mathbb{R}_+$. Let $m = \inf_{[-1,0]} \phi(s), M = \sup_{[-1,0]} \phi(s)$. Then $[m, M] \subset J_0$. J_0 being the domain of immediate attraction of $\overline{x}$ implies that there exists a closed interval I such that $[m, M] \subseteq I \subset J_0, F(I) \subset I$, and $\cap_{n \geq 0} F^n(I) = \overline{x}$. The repeated application of the contraction property (Lemma 2) shows that there exists a sequence $T_n = T_n(\phi) \uparrow \infty$ such that $x(t) \in F^n(I)\ \forall t \geq T_n$.

Corollary 1 *Assume (H0)-(H2) and that $\overline{x}$ is the globally attracting fixed point of the map F : $\lim_{n\to\infty} F^n(x) = \overline{x}\ \forall x \in \mathbb{R}_+ \setminus \{0\}$. Then for every initial function $\phi(s) \in C([-1, 0], \mathbb{R}_+)$ with $\phi(s) > 0\ \forall s \in [-1, 0]$ the corresponding solution $x(t)$ has the property $\lim_{t\to\infty} x(t) = \overline{x}$.*

We indicate next a couple of simple conditions under which the fixed point $\overline{x}$ of the map F is globally attracting in $\mathbb{R} \setminus \{0\}$. In view of Corollary 1 the constant solution $x(t) = \overline{x}$ of equation (2.1) (resp. (1.1)) is globally asymptotically stable for all $\mu > 0$ (resp. $h > 0$).

In view of the Sharkovsky cycle coexistence ordering [21] the most general condition for $\overline{x}$ to be globally attracting is that

(C) The second iteration F^2 of the map F does not have a fixed point in $\mathbb{R}_+ \setminus \{0\}$ other than $\overline{x}$, and $\overline{x}$ is locally attracting.

Some partial cases that imply (C) are the following

(C1) $f(x) > g(x)$ and $f(x) < g(\overline{x})\ \forall x \in (0, \overline{x})$;

(C2) $f(x) < g(x)$ and $f(x) > g(\overline{x})\ \forall x \in (\overline{x}, \infty)$;

(C3) $F(x)$ is unimodal or strictly monotone, has the negative Schwarz derivative, $SF(x) < 0$, outside the critical point, and $|F'(\overline{x})| \leq 1$ [21].

2.3 Absolute global stability: a necessary condition

In the case when f_i, g_j are all smooth functions, it is rather easy to indicate a simple necessary condition for the absolute global stability of Eq. (1.1). Indeed, in this case the equilibrium is also locally asymptotically stable for all $h > 0$, so that the variational equation

$$\begin{aligned} y'(t) &= \begin{vmatrix} f_1(\overline{x}) & g_1'(\overline{x}) \\ f_2(\overline{x}) & g_2'(\overline{x}) \end{vmatrix} y(t) + \begin{vmatrix} f_1'(\overline{x}) & g_1(\overline{x}) \\ f_2'(\overline{x}) & g_2(\overline{x}) \end{vmatrix} y(t-h) \\ &= Ay(t) + By(t-h) \end{aligned}$$

along $x(t) \equiv \overline{x}$ should be stable for all $h \geq 0$ (in our special case). It is a well known fact (see, for example, [8]) that this is possible only when

$$A \leq 0 \quad \text{and} \quad -A \geq |B|. \tag{2.4}$$

The first inequality in (2.4) is equivalent to $g'(\overline{x}) \geq 0$ and is satisfied automatically in view of (H1). Furthermore, if $g'(\overline{x}) = 0$ and Eq. (1.1) is absolutely stable, then, due to the second condition in (2.4), we have $f_1'(\overline{x})g_2(\overline{x}) - f_2'(\overline{x})g_1(\overline{x}) = 0$ implying $f'(\overline{x}) = 0$. Thus, in the case $g'(\overline{x}) = 0$, we obtain the following necessary condition for the global absolute stability: $f'(\overline{x}) = 0$. Suppose now that $g'(\overline{x}) > 0$. In this case, as an elementary calculation shows, the second condition in (2.4) is equivalent to $|F'(\overline{x})| = |f'(\overline{x})/g'(\overline{x})| \leq 1$. By summing up the above, we have

Lemma 3 *If the equilibrium $x = \overline{x}$ of Eq. (1.1) is globally absolutely stable then $|f'(\overline{x})| \leq g'(\overline{x})$.*

3 Delay dependent conditions for global stability

In this section, we are assuming all three hypotheses (H0)-(H2). Moreover, to simplify the exposition, we suppose here that f_i are continuous and g_j are Lipschitz continuous so that we have the existence and uniqueness of solutions for Eq. (1.1) for all $t \geq 0\,(i = 1, 2)$.

3.1 General global stability theorem

From Subsections 2.2 and 2.3, we know that in some cases condition $|F'(\overline{x})| \leq 1$ is necessary and sufficient for the absolute global stability. In this subsection we assume that F is differentiable at $\overline{x}$ and $|F'(\overline{x})| > 1$, that is, $-f'(\overline{x}) > g'(\overline{x}) > 0$.

Let $x(t)$ be a solution of (1.1). In view of Theorem 2, one has $0 < \liminf_{t\to+\infty} x(t) = m \leq \limsup_{t\to+\infty} x(t) = M < +\infty$.

Now we prove several useful lemmas:

Lemma 4 *$F([m, M]) \supseteq [m, M]$.*

Proof. If $x(t)$ is monotone then, by (1.1), there exists the finite limit $x'(+\infty)$ which should be equal to 0. The latter implies $m = M = \overline{x}$ and lemma is proved. The same argument works when we assume $m = M$. Hence, the only case we are interested in is $m < M$. In this case $x(t)$ oscillates. We can find sequences $x(t_n) \to m, x(s_n) \to M$ such that $x'(t_n) = x'(s_n) = 0$. Using (1.1), we get immediately that $x(t_n) = F(x(t_n - h))$ and $x(s_n) = F(x(s_n - h))$ so that $m, M \in F([m, M])$. Since $F([m, M])$ is a connected interval, Lemma 4 is proved.

Corollary 2 *Suppose that $x(t)$ is not converging to $\overline{x}$. Then $x(t)$ oscillates around $\overline{x}$.*

Proof. Indeed, in this case $m < M$ and, by Lemma 4, there exists at least one fixed point c of F in $[m, M]$. Since F has a unique equilibrium, $c = \overline{x}$.

Lemma 5 *Let $x : \mathbb{R} \to \mathbb{R}_+$ be a solution of (1.1) and $M = x(\theta) > \overline{x}$ be its global maximum. Then $x(\theta - h) \leq \overline{x}$. Likewise, if $m = x(\theta_1) < \overline{x}$ is its global minimum, then $x(\theta_1 - h) \geq \overline{x}$.*

Proof. Assume that $M = x(\theta) > \overline{x}$ and $x(\theta - h) > \overline{x}$. Then, by (H2), $f(x(\theta - h)) < g(x(\theta - h)$ and therefore, by (H1), $F(x(\theta - h)) < x(\theta - h)$. On the other hand, since $x'(\theta) = 0$ we have that $f(x(\theta - h)) = g(x(\theta))$, implying that $M = x(\theta) = F(x(\theta - h)) < x(\theta - h)$, a contradiction with the assumption that M is the global maximum.

The proof for m is completely analogous.

According to Corollary 2, we know that every solution $x : \mathbb{R}_+ \to \mathbb{R}_+$ of Eq. (1.1) is either oscillating around the steady state $\overline{x}$ or monotonically converging to it.

Next, if $x(t)$ is a solution to (1.1), then the trajectory $x_t : \mathbb{R}_+ \to C$ where $x_t(s) = x(t+s), s \in [-h, 0]$ can be considered. It is easy to see that for every $\phi \in \omega(x) = \cap_{t\geq 0}\overline{\{x_r, r \geq t\}} \subset C$ there exists a solution $x(s,\phi) : \mathbb{R} \to \mathbb{R}_+$ such that $\phi(s) = x(s,\phi),\ s \in [-h, 0]$. Obviously, we have for all $\phi \in \omega(x)$ that

$$\liminf_{t\to+\infty} x(t) = m \leq x(s,\phi) \leq M = \limsup_{t\to+\infty} x(t),$$

while $M = x(q_1, \psi), m = x(q_2, \chi)$ for some $\psi, \chi \in C$ and $q_1, q_2 > 0$. Moreover, by Lemma 5, we may assume that $q_i \leq h,\ i = 1, 2$, and $x(0,\psi) = x(0,\chi) = \overline{x}$ with $x(t,\psi) \in (\overline{x}, M)$ for all $t \in (0, q_1)$, $\ x(t,\chi) \in (m, \overline{x})$ for all $t \in (0, q_2)$. Now, since

$$x'(t,\psi) = \begin{vmatrix} f_1(x(t-h,\psi)) & g_1(x(t,\psi)) \\ f_2(x(t-h,\psi)) & g_2(x(t,\psi)) \end{vmatrix} \leq \begin{vmatrix} \max_{u\in[m,M]} f_1(u) & g_1(x(t,\psi)) \\ \min_{u\in[m,M]} f_2(u) & g_2(x(t,\psi)) \end{vmatrix},$$

by the standard comparison results (see, e.g. [22, Theorem 5.III]), we conclude that $x(t,\psi) \leq z(t, m, M), t \geq 0$, where $z(t, m, M)$ solves the initial value problem

$$z'(t) = F_+(z(t), m, M), z(0) = \overline{x}, \text{ where } F_+(z, m, M) = \begin{vmatrix} \max_{u\in[m,M]} f_1(u) & g_1(z) \\ \min_{u\in[m,M]} f_2(u) & g_2(z) \end{vmatrix}. \tag{3.1}$$

Furthermore, since $-f'(\overline{x}) > g'(\overline{x}) \geq 0$, we have that $g(z) < \max_{u\in[m,M]} f(u) \leq \max_{u\in[m,M]} f_1(u)/ \min_{u\in[m,M]} f_2(u)$ for all z in some interval $[\overline{x}, U)$.

Thus $F_+(z, m, M) > 0$ for all $z \in [\overline{x}, U)$, and therefore function

$$\Pi(\zeta) = \int_{\overline{x}}^{\zeta} \frac{dz}{F_+(z, m, M)}$$

is well defined for $\zeta \in [\overline{x}, U)$.

Let $\zeta_* \geq U$ be the smallest zero of $F_+(z, m, M)$, for fixed m, M. Notice that such zero $\zeta_* \geq U$ always exists, for otherwise $F_+(z, m, M)/g_1(z) > 0$ for all $z > \overline{x}$, while $\lim_{z\to+\infty} F_+(z, m, M)/g_1(z) = -\min_{u\in[m,M]} f_2(u) < 0$. Since $F_+(z, m, M) > 0$ on $(\overline{x}, \zeta_*)$ and since $F_+(z, m, M)$ is smooth at $z = \zeta_*$, we find that $\Pi(\zeta_* - 0) = +\infty$. This property of $\Pi(\zeta)$ is sufficient to justify the reasoning that follows.

Integrating (3.1), we find that $\Psi(m, M) = z(q, m, M) \geq M$ satisfies the equation

$$\int_{\overline{x}}^{\Psi} \frac{dz}{F_+(z, m, M)} = q \leq h.$$

Therefore, if we define $\Phi_+ = \Phi_+(m, M) > \overline{x}$ as the unique solution of

$$\int_{\overline{x}}^{\Phi_+} \frac{dz}{F_+(z, m, M)} = h,$$

on the interval $[\overline{x}, \zeta_*)$, we obtain that $\Phi_+(m, M) \geq \Psi(m, M) \geq M$. Likewise, an analogous consideration of $x(t,\chi)$ shows that $\Phi_-(m, M) \leq m$, where Φ_- is defined by

$$\int_{\overline{x}}^{\Phi_-} \frac{dz}{F_-(z, m, M)} = h, \quad \text{where } F_-(z, m, M) = \begin{vmatrix} \min_{u\in[m,M]} f_1(u) & g_1(z) \\ \max_{u\in[m,M]} f_2(u) & g_2(z) \end{vmatrix}.$$

Summarizing the above, we get the following statement:

Theorem 4 *Suppose that the system of two inequalities*

$$\Phi_+(a,b) \geq b, \quad \Phi_-(a,b) \leq a$$

does not have a solution (a,b) *such that* $0 < a < \overline{x} < b$. *Then the steady state* $x(t) = \overline{x}$ *of Eq. (1.1) is globally attracting.*

As a simple application of Theorem 4, we prove here the following global asymptotic stability condition for Eq. (1.5), which is a partial case of Eq. (1.1) with $f_1 = f, g_1 = g, f_2 = g_2 = 1$ (announced in [9]):

Corollary 3 *Suppose that* $x = \overline{x}$ *is the globally attracting fixed point of the map*

$$G(x) := \frac{1-e^{-\alpha h}}{\alpha}\ f(x) + \left[\overline{x} - \frac{1-e^{-\alpha h}}{\alpha}\ f(\overline{x})\right], \tag{3.2}$$

where $\alpha := \inf\{\frac{g(x)-g(\overline{x})}{x-\overline{x}}, x > 0\}$ *is assumed to be positive. Then the constant solution* $x(t) = \overline{x}$ *of Eq. (1.5) is globally asymptotically stable.*

Notice that for $g(x)$ differentiable the quantity $\beta := \inf\{g'(x), x > 0\}$ can be used as an approximation to α.

Proof. Take an arbitrary solution $x(t)$ of Eq. (1.5) and consider the real values

$$0 < m = \liminf_{t\to+\infty} x(t) \leq \overline{x} \leq M = \limsup_{t\to+\infty} x(t).$$

Following the notations in the proof of Theorem 4, we have, in view of the definition of α,

$$\begin{aligned} h &= \int_{\overline{x}}^{\Phi_+} \frac{dz}{F_+(z,m,M)} = \int_{\overline{x}}^{\Phi_+} \frac{dz}{-g(z) + \max_{u\in[m,M]} f(u)} \\ &\geq \int_{\overline{x}}^{\Phi_+} \frac{dz}{-\alpha(z-\overline{x}) - g(\overline{x}) + \max_{u\in[m,M]} f(u)}. \end{aligned}$$

Integrating the last expression, we obtain:

$$h \geq \frac{-1}{\alpha} \ln \frac{\alpha(\Phi_+ - \overline{x}) + g(\overline{x}) - \max_{u\in[m,M]} f(u)}{g(\overline{x}) - \max_{u\in[m,M]} f(u)}.$$

Hence, since $g(\overline{x}) = f(\overline{x}) < \max_{u\in[m,M]} f(u)$, we get

$$\Phi_+(m,M) \leq \overline{x} + \frac{1-e^{-\alpha h}}{\alpha}\left(\max_{u\in[m,M]} f(u) - f(\overline{x})\right) = \max_{u\in[m,M]} G(u),$$

where G is defined by (3.2).

Analogously, we can prove that $\Phi_-(m,M) \geq \min_{u\in[m,M]} G(u)$. Now, if (m,M) is such that $0 < m \leq \overline{x} \leq M$, $\Phi_+(m,M) \geq M$, and $\Phi_-(m,M) \leq m$, then it follows that $M \leq \max_{u\in[m,M]} G(u)$ and $m \geq \min_{u\in[m,M]} G(u)$, implying that $[m,M] \subset G([m,M])$. Finally, since $\overline{x}$ is a globally attracting fixed point of G, we conclude that $m = \overline{x} = M$, and the proof of the corollary follows from Theorem 4.

3.2 Proof of Theorem 1

In this subsection we prove Theorem 1 stated in Section 1 as a consequence of Theorem 4.

Let us observe that Eq. (1.4) is a particular case of (1.1), with $F(x) = 1/\varphi(x)$, $x > 0$. Hence F is strictly decreasing.

From the properties of Schwarz derivative, we have that $(S\varphi)(x) < 0$ implies $(SF)(x) < 0$ for all $x > 0$. Hence the global attractivity of $\overline{x}$ in the case $|F'(\overline{x})| = \overline{x}^2\varphi'(\overline{x}) \leq 1$ is a direct consequence of Corollary 1 and the remarks below it.

Therefore, we can assume that $|F'(\overline{x})| > 1$.

Take an arbitrary solution $x(t)$ of Eq. (1.4) and consider the real values

$$0 < m = \liminf_{t\to+\infty} x(t) \leq \overline{x} \leq M = \limsup_{t\to+\infty} x(t).$$

By using the notations in the proof of Theorem 4, we have

$$\Phi_+(m, M) = G(F(m)) \quad , \quad \Phi_-(m, M) = G(F(M))\,,$$

where $G(x) = x - e^{-h/x}(x - \overline{x})$.

Hence Theorem 1 will be proved if we show that the unique solution of the system of inequalities

$$G(F(M)) \leq m\ ,\ G(F(m)) \geq M\ , 0 < m \leq \overline{x} \leq M$$

is $m = M = \overline{x}$.

Remark 3.1 *Notice that $G \circ F : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly decreasing map (continued continuously at $x = 0$ if $\varphi(0) = 0$, since $G(+\infty) = \overline{x} + h$). These relations imply immediately the following estimations for the limit values of any solution x of Eq. (1.4):*

$$G(F(\overline{x} + h)) \leq \liminf_{t\to+\infty} x(t) = m \leq \overline{x} \leq M = \limsup_{t\to+\infty} x(t) \leq G(+\infty) = \overline{x} + h.$$

To finish the proof of Theorem 1, we have to examine the behavior of one-dimensional map $G \circ F$. Numerical evaluations show that the Schwarz derivative of $G \circ F$ is negative; however, we were not able to find an analytic proof of this (unfortunately, $(SG)(x)$ can be positive for small values of x so that we cannot apply the well-known argument that $S(G \circ F)$ is negative if both SG and SF are negative). To avoid the necessity to check every time that $S(G \circ F)(y)$ is negative, we will approximate G by a rational map.

Set

$$a = G'(\overline{x}) = 1 - e^{-h/\overline{x}} \quad , \quad b = G''(\overline{x}) = \frac{-2he^{-h/\overline{x}}}{\overline{x}^2},$$

and define the rational function

$$Q(x) = \overline{x} + \frac{a(x - \overline{x})}{1 - (b/2a)(x - \overline{x})} = \overline{x} + \frac{(1 - \exp(-h/\overline{x}))(x - \overline{x})}{1 + \frac{h\exp(-h/\overline{x})(x-\overline{x})}{(1-\exp(-h/\overline{x}))\overline{x}^2}}.$$

This function satisfies $Q(\overline{x}) = G(\overline{x}) = \overline{x}$, $Q'(\overline{x}) = G'(\overline{x}) = a$, $Q''(\overline{x}) = G''(\overline{x}) = b$. Moreover, Q is well defined for all $x \geq 0$ since $1 - (b/2a)(x - \overline{x}) = 0$ only for $x = \overline{x} - h^{-1}\overline{x}^2e^{h/\overline{x}}$, which is negative due to the relation $e^{h/\overline{x}} > 1 + h/\overline{x}$.

Next, the following lemma will allow us to work with Q instead of G:

Lemma 6 *Q is strictly increasing on $\mathbb{R}_+$. Furthermore, $Q(x) > G(x)$ for all $x > \overline{x}$ and $Q(x) < G(x)$ for all $0 < x < \overline{x}$.*

Proof. Since $Q'(x) = a(1-(b/2a)(x-\overline{x}))^{-2} > 0$, it follows that Q is strictly increasing on $\mathbb{R}_+$.

Now, for positive $x \neq \overline{x}$, the inequality $(Q(x) - G(x))(x - \overline{x}) > 0$ is equivalent to

$$1 + \frac{h\exp(-h/\overline{x})(x-\overline{x})}{(1-\exp(-h/\overline{x}))\overline{x}^2} < \frac{1-\exp(-h/\overline{x})}{1-\exp(-h/x)},$$

which is true since we have, for some θ between $\overline{x}$ and x, that

$$\frac{1-\exp(-h/\overline{x})}{1-\exp(-h/x)} = 1 + \frac{h\exp(-h/\overline{x})(x-\overline{x})}{(1-\exp(-h/\overline{x}))\overline{x}^2} + T(\theta,\overline{x},h)(x-\overline{x})^2/2,$$

where

$$T(\theta,\overline{x},h) = \frac{h\exp(-h/\theta)(1-\exp(-h/\overline{x}))}{(1-\exp(-h/\theta))^3\theta^4}\left((2\theta+h)\exp(-h/\theta)+h-2\theta\right) \geq 0.$$

Lemma 6 implies immediately that $\Phi = Q \circ F$ is well-defined and strictly decreasing on $\mathbb{R}_+$. Notice that $S\Phi = S(Q\circ F)(x) = (SF)(x) < 0$ and that $\Phi'(\overline{x}) = (Q\circ F)'(\overline{x}) = (G\circ F)'(\overline{x})$. Moreover, in the case of our interest Φ maps $[0, \Phi(0)]$ into itself:

Lemma 7 *If $|\Phi'(\overline{x})| < 1$, then $\Phi^2(0) = \Phi\circ\Phi(0) > 0$.*

Proof. Suppose that $\Phi^2(0) \leq 0$ and take the maximal $x_1 < \overline{x}$ such that $\Phi^2(x_1) = 0$. Since $(\Phi\circ\Phi)'(\overline{x}) < 1$ it follows that $\Phi\circ\Phi$ has at least one fixed point on $(x_1, \overline{x})$. Let x_2 be the largest fixed point of $\Phi\circ\Phi$ on $(x_1,\overline{x})$. Obviously, $\Phi(x_2)$ is also a fixed point of $\Phi\circ\Phi$ so that the interval $I = [x_2, \Phi(x_2)]$ is invariant under the action of Φ^2. Since both endpoints of I and $\overline{x} \in I$ are all fixed points by Φ^2, and $(\Phi^2)'(\overline{x}) < 1$, we obtain a contradiction with the assumption $S\Phi^2 < 0$.

From previous Lemmas, we obtain that $M \leq G\circ F(m) \leq Q\circ F(m) = \Phi(m) \leq \Phi(0)$ and $m \geq G\circ F(M) \geq Q\circ F(M) = \Phi(M)$. Hence $[m, M] \subseteq [0, \Phi(0)]$ and $[m, M] \subseteq \Phi[m, M]$. Finally, since $|\Phi'(\overline{x})| = |G'(\overline{x})F'(\overline{x})| = \overline{x}^2(1-\exp(-h/\overline{x}))\varphi'(\overline{x}) < 1$, and $S\Phi < 0$, we conclude that $m = \overline{x} = M$. This completes the proof of Theorem 1.

3.3 An example

In this subsection we apply our results to obtain some global stability conditions for a delay equation arising in respiratory dynamics.

The following differential delay equation

$$x'(t) = \begin{vmatrix} \lambda & x(t) \\ \frac{\mu x^n(t-h)}{\sigma^n + x^n(t-h)} & 1 \end{vmatrix}, \quad n, h, \sigma, \mu > 0, \tag{3.3}$$

was proposed by Mackey and Glass (see [11, 18, 20]) to model dynamics of respiratory systems. The following result was established in [12, p. 158]:

Proposition 5 *Let $x(t) \equiv x^*$ be the unique positive steady state of (3.3). Assume that one of the following two conditions holds: (i) $n \le 1$ and $\mu < 2x^*\lambda\sigma^{-1}$; (ii) $n > 1$, $n\lambda\sigma^{-1} > 0.25\mu(n+1)^{(n+1)/n}(n-1)^{(n-1)/n}$. Then it is absolutely globally asymptotically stable. If $(\lambda\sigma^{-1}+\mu(n+1)^{(n+1)/n}(n-1)^{(n-1)/n}/(4n))\mu\lambda\sigma^{-1}h < 1$ and $n > 1$, then $x(t) \equiv x^*$ is globally asymptotically stable.*

A different and rather more complicated sufficient condition for the global asymptotic stability of the unique positive equilibrium of (3.3) was also proved in [4, Theorem 3.1]. Our analysis shows that Proposition 5 provides better conditions for the stability compared with those given in [4] (also see a numerical comparison at the end of this paper).

Applying Corollary 1, we get the following improvement of the absolute global stability part of Proposition 5:

Theorem 5 *If (i) $n \le 1$ or (ii) $n > 1$ and $n\lambda\sigma^{-1} \ge \mu(n-1)^{(n+1)/n}$, then the steady state $x(t) \equiv x^*$ of Eq. (3.3) is globally attracting for every $h > 0$.*

Proof. We have that $g(x) = x$ and $F(x) = f(x) = \lambda\mu^{-1}(\sigma^n + x^n)/x^n$.

(i) The equation $F(F(a)) = a$ determines all 2-periodic points of $F : (0,\infty) \to (0,\infty)$. Since $F(x) > \lambda\mu^{-1}$ for all $x > 0$ we obtain that necessarily $a > \lambda\mu^{-1}$. Using the strict monotonicity of F, we can conclude that $F(a) = F^{-1}(a)$, which implies immediately

$$p(a) = \left(\frac{\lambda}{\mu}\right)^n (\sigma^n + a^n)^n = \frac{\lambda\sigma^n a^{n^2}}{\mu a - \lambda} = q(a).$$

This equation has exactly one solution a for every $n \in (0,1]$ since $p(a)$ increases in a and $q(a)$ is decreasing. Since F has only one periodic point (which is exactly the fixed point x^*) it must attract all trajectories of F. An application of Corollary 1 ends the consideration of this case.

(ii) Let now $n > 1$. Since F is strictly decreasing and has negative Schwarz derivative: $(SF)(x) = F'''(x)/F'(x) - (3/2)(F''(x)/F'(x))^2 = (-n^2+1)/(2x^2) < 0$, we can apply the well-known Singer theorem to conclude that the local attractivity of the equilibrium x^* of the map $F : (0,\infty) \to (0,\infty)$ implies its global attractivity. Now, a direct verification shows that $|F'(x^*)| \le 1$ if and only if $x^* \ge (n\lambda\sigma^n/\mu)^{1/(n+1)}$. Fix now σ and consider the solution x^* of the determining equation $\lambda\mu^{-1}(\sigma^n + x^n) = x^{n+1}$ as a function of the parameter $\lambda\mu^{-1}$: it is clear that $x^*(\lambda\mu^{-1})$ is strictly decreasing in $\lambda\mu^{-1}$ while $(n\lambda\sigma^n/\mu)^{1/(n+1)}$ is strictly increasing in the same variable. This implies that there exists $\lambda_0\mu_0^{-1}$ such that $x^*(\lambda\mu^{-1}) \ge (n\lambda\sigma^n/\mu)^{1/(n+1)}$ for all $\lambda\mu^{-1} \ge \lambda_0\mu_0^{-1}$. Moreover, when $\lambda\mu^{-1} = \lambda_0\mu_0^{-1}$, we obtain $x^*(\lambda_0\mu_0^{-1}) = (n\lambda_0\sigma^n/\mu_0)^{1/(n+1)}$. Finally, we get (using the determining equation) that $\lambda_0\mu_0^{-1} = \sigma n^{-1}(n-1)^{n+1/n}$.

Next, we apply Theorem 1 to establish delay dependent global stability conditions for Eq. (3.3). First, we simplify this equation by applying the change of variable $x = \lambda y$ to get

$$y'(t) = \begin{vmatrix} 1 & y(t) \\ \frac{\mu y^n(t-h)}{\sigma_1^n + y^n(t-h)} & 1 \end{vmatrix}, \quad n, h, \mu > 0, \tag{3.4}$$

with $\sigma_1 = \sigma/\lambda > 0$.

Let $\overline{y}$ be the unique positive equilibrium of Eq. (3.4). As a direct consequence of Theorem 1, we have the following easily verifiable delay dependent stability condition:

Corollary 4 *Assume that* $(1-\exp(-h/\overline{y}))n\sigma_1^n/(\mu\overline{y}^{n+1}) < 1$. *Then every solution of (3.4) converges to* $\overline{y}$.

To have a more clear idea about how our delay dependent stability condition given by Corollary 4 compares with others, consider Eq. (3.3) with $\lambda = 1, n = \sigma = \mu = 2$. In this case, Theorem 3.1 in [4] provides the global attractivity for all $h < 0.230854...$, while Proposition 5 improves this condition up to $h < (0.5 + 3\sqrt{3}/4)^{-1} = 0.555....$ Now, we have that $\overline{y} = (1 + (217 + 12\sqrt{327})^{1/3} + (217 + 12\sqrt{327})^{-1/3})/6 = 1.450540170...$ and that $4(1 - \exp(-h/\overline{y})) < \overline{y}^3$ for all $h < -\overline{y}\ln(1 - \overline{y}^3/4) = 2.088...$ We can apply Corollary 4 to derive the global stability of $y(t) \equiv \overline{y}$ for all $h < 2.088....$ Finally, the local analysis of the steady state of Eq. (3.3) shows that it losses the local stability when $h > 4.18....$

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