Journal of Difference Equations and Applications,
Vol. 12, No. 12, December 2006, 1201–1212

Periodicity on discrete dynamical systems generated by a class of rational mappings

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(Received 6 June 2006; in final form 3 August 2006)

We address the problem of global periodicity in discrete dynamical systems generated by rational maps in $\mathbb{R}^k$ or $\mathbb{C}^k$. Our main results show that for a wide family of such maps, this problem may be reduced to the analysis of a related matrix equation. We use this fact to estimate the number of possible minimal periods in globally periodic maps of this class when all the involved coefficients are rational.

Keywords: Global periodicity; Discrete dynamical system; Periodic point; Rational map; Euler's phi-function

Mathematics Subject Classification (2000): 39B12; 39A11; 39B42

1. Introduction

A classical problem in functional equations consists in finding primitive iterative K-roots of the identity map, that is, functions φ such that $\varphi^K = \mathrm{Id}$ and $\varphi^m \neq \mathrm{Id}$ for all $m = 1, \ldots, K-1$. From the dynamical point of view, a map $F : \mathcal{U} \to \mathcal{U}$ generates a discrete dynamical system on $\mathcal{U}$ defined by the recurrence

$$x_{m+1} = F(x_m), \quad m = 0, 1, \ldots \tag{1.1}$$

Thus, saying that $F^K = \mathrm{Id}$ is equivalent to saying that for all initial condition x_0, the corresponding orbit $\{x_m\}$ of equation (1.1) is K-periodic, that is, $x_{m+K} = x_m$ for all $m \geq 0$. This is a very interesting feature in the dynamics which has recently attracted more and more attention (see, e.g. the monographs [8,11]). In this case, we say that the recurrence (1.1) is globally K-periodic.

An important example in which global periodicity may appear is provided by real or complex Möbius transformations

$$F(x) = \frac{ax + b}{cx + d}, \tag{1.2}$$

where $a, b, c, d \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Different aspects of the problem for this class of maps were considered in Refs. [4,6,10].

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Journal of Difference Equations and Applications
ISSN 1023-6198 print/ISSN 1563-5120 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10236190600949782

Other recent papers are focused in a generalization of the first order recurrence generated by equation (1.2) to higher order rational difference equations in the form

$$x_{m+n} = \frac{a_0 + a_1 x_m + \cdots + a_n x_{m+n-1}}{b_0 + b_1 x_m + \cdots + b_n x_{m+n-1}} = f(x_m, \ldots, x_{m+n-1}), \quad m = 0, 1, \ldots, \tag{1.3}$$

where $n \geq 1$ is a fixed integer, and the coefficients $a_i, b_i, i = 0, 1, \ldots, n$, are either real or complex. For $n = 2$, we refer to [11], and, for general n, some results were obtained in Refs. [1,2,5,14]. With equation (1.3), one can associate a dynamical system given by the following map $F : D \subset \mathbb{K}^n \to D$:

$$F(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n)), \tag{1.4}$$

in such a way that equation (1.3) is globally K-periodic if and only if $F^K = \mathrm{Id}$ in D. However, there are other interesting rational dynamical systems which exhibit the property of global periodicity and do not come from difference equations. For example, the flow at time $\pi/2$ of the 2π-isochronous planar system (see Ref. [7] and references therein)

$$x'(t) = -y(t) + x^2(t); \quad y'(t) = x(t)(1 + y(t))$$

is given by the map

$$F(x, y) = \left(\frac{-y}{1 - x + y}, \frac{x}{1 - x + y} \right).$$

It is clear by its construction that $F^4 = \mathrm{Id}$, that is, F is 4-periodic. See more discussions in Ref. [3].

In this paper, we deal with rational equations of this type. More precisely, we will consider a map $F(x) = (F_1(x), \ldots, F_n(x))$ defined on some open subset $\mathcal{U} \subset \mathbb{K}^n$, of the form

$$F_i(x_1, x_2, \ldots, x_n) = \frac{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + c_i}{b_1 x_1 + b_2 x_2 + \cdots + b_n x_n + d}, \quad i = 1, 2, \ldots, n, \tag{1.5}$$

where $a_{ij}, b_i, c_i, d \in \mathbb{K}$ and $|b_1| + \cdots + |b_n| + |d| \neq 0$. Our main results show that the problem of global periodicity for equation (1.5) may be reduced to the analysis of a related matrix equation. We use this fact to estimate the number of possible minimal periods of globally periodic maps in the form (1.5) when all the involved coefficients are rational, extending the results in Ref. [6] for the scalar Möbius map (1.2). As far as we know, our results in this direction are new even for linear maps.

The paper is structured as follows: in Section 2, we show that any map of the form (1.5) may be defined as a projection of a linear map acting on an open subset of $\mathbb{K}^n$. This fact is used in Section 3 to characterize the periodic points of F, and to give a criterion for the global periodicity of this family of maps. Special attention is paid to the case in which all coefficients in equation (1.5) are rational numbers. Finally, in Section 4 we address the problem of eventual global periodicity for equation (1.5): namely, we find necessary and sufficient conditions to ensure that F is not K-periodic but there exists $m \in \mathbb{N}$ such that F^m is K-periodic.

2. Preliminaries and first results

From now on, $\mathbb{K}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{M}_{n\times n}(\mathbb{K})$ denotes the set of square matrices of size $n \times n$ with entries in $\mathbb{K}$. Let $\mathcal{U}$ be an open subset of $\mathbb{K}^n$ and consider a map $F : \mathcal{U} \to \mathcal{U}$.

As usual, for each $m \in \mathbb{N}$, F^m means the corresponding power under composition, this is to say, $F^m = \overbrace{F \circ \cdots \circ F}^{m \text{ times}}$.

Definition 2.1. *We will say that point $x \in \mathcal{U}$ is K-periodic if $F^K(x) = x$. As mentioned in the introduction, the map F is said to be globally periodic if there exists $K \in \mathbb{N}$ such that $F^K = \mathrm{Id}_U$, where* Id *stands for the identity map. If, in addition, $F^m \neq \mathrm{Id}_U$ for all $m = 1, 2, \ldots, K-1$, then we say that K is the minimal period of F.*

Definition 2.2. *Let $F(x) = (F_1(x), \ldots, F_n(x))$ be defined by equation* (1.5) *and let us consider the open set*

$$\mathcal{P}^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{K}^{n+1} : x_{n+1} \neq 0\right\}.$$

We define the mappings $q : \mathcal{P}^n \to \mathbb{K}^n$ and $\ell : \mathbb{K}^n \to \mathcal{P}^n$ respectively by

$$q(x_1, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right); \quad \ell(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 1).$$

Functions q and ℓ will play an important role in our exposition. Next, we state some basic properties of them for convenience of the reader. The proof is straightforward, so we omit it.

Lemma 2.1. *Using the notations of Definition* 2.2*, we have:*

(a) $q(\ell(x)) = x$ *for all* $x \in \mathbb{K}^n$.
(b) $\ell(q(x)) = (1/x_{n+1})x$ *for all* $x = (x_1, \ldots, x_{n+1}) \in \mathcal{P}^n$.
(c) *If* $x, y \in \mathcal{P}^n$, *then* $q(x) = q(y) \Leftrightarrow x = \lambda y$ *for some* $\lambda \neq 0$.

From items (b) and (c) in Lemma 2.1 we easily get the following corollary:

Corollary 2.2. *If A is an $(n+1) \times (n+1)$ matrix, then $q(A\ell(q(x))) = q(Ax)$ for all $x \in \mathcal{P}^n$.*

Our next result characterizes the powers of F in terms of the corresponding powers of a related square matrix A.

Proposition 2.3. *Let F be the rational map given by equation* (1.5)*, and define $A \in \mathcal{M}_{n+1 \times n+1}(\mathbb{K})$ by*

$$A = \left(\begin{array}{c|c} M & C^t \\ \hline B & d \end{array}\right),$$

where $M = (a_{ij}) \in \mathcal{M}_{n \times n}(\mathbb{K})$, and $B = (b_1, \ldots, b_n), C = (c_1, \ldots, c_n) \in \mathbb{K}^n$.

Then $F^K(x) = q(A^K \ell(x))$ holds for every integer $K \geq 1$ and every $x \in \mathbb{K}^n$ for which $F^K(x)$ is defined.

Proof. We will prove the result by induction on K. It is straightforward to verify the result for $K = 1$, therefore let us suppose that $F^{K-1} = qA^{K-1}\ell$. Using Corollary 2.2, we have, for all $x \in \mathbb{K}^n$ such that $F^K(x)$ exists, that

$$F^K(x) = F(F^{K-1}(x)) = q(A\ell(q(A^{K-1}\ell(x)))) = q(AA^{K-1}\ell(x)) = q(A^K\ell(x)),$$

which finishes the proof. □

3. Global periodicity of *F*

In this section, we present our main results on the periodic points and the global periodicity problem for rational maps defined by equation (1.5).

A natural question is how large is the set of points $x \in \mathbb{K}^n$ for which the orbit $\{F^m(x) : m \geq 0\}$ cannot be defined completely. Sometimes this set is called *forbidden set* (see, e.g. [11]), and we will denote it as $FS(F)$. Notice that

$$FS(F) = \{x \in \mathbb{K}^n : A^m\ell(x) \notin \mathcal{P}^n, \quad \text{for some } m \geq 1\}, \tag{3.1}$$

and thus it is a sequence of hyperplanes in $\mathbb{K}^n$.

Furthermore, if F is K-periodic, then the forbidden set consists only of at most $(K - 1)$ hyperplanes in $\mathbb{K}^n$ given by

$$FS(F) = \bigcup_{m=1}^{K-1} \{x \in \mathbb{K}^n : A^m\ell(x) \notin \mathcal{P}^n\}.$$

3.1 Periodic points and global periodicity

Our main result is a criterion for the global periodicity of F. Here, I denotes the identity matrix.

THEOREM 3.1. *Let $F(x) = (F_1(x), \ldots, F_n(x))$ be defined by equation* (1.5) *and let A be a matrix such that $F = qA\ell$.*

(a) *A point $x \in \mathbb{K}^n$ is K-periodic if and only if $\ell(x)$ is an eigenvector of A^K* (*necessarily associated to a non-zero eigenvalue*).
(b) *F is globally K-periodic if and only if $A^K = \lambda I$ for some $\lambda \neq 0$.*

Proof. According to Lemma 2.1 (a) and Proposition 2.3, $F^K(x) = x$ is equivalent to $q(A^K\ell(x)) = x = q(\ell(x))$. The statement of part (a) follows at once from Lemma 2.1 (c).

In order to prove (b), for each $N \in \mathbb{N}\backslash\{0\}$, let us consider the domain of definition of F^N, given by the open set

$$\mathcal{U}_N = \{x \in \mathbb{K}^n : A^j\ell(x) \in \mathcal{P}^n, \quad \text{for all } j = 1, \ldots, N\}.$$

If $A^K = \lambda I$ for some $\lambda \neq 0$, then it follows easily from Proposition 2.3 that F is globally K-periodic. Conversely, if F is a K-periodic map then $qA^K\ell = q\ell$ on $\mathcal{U}_K$ and hence for each

$x \in \mathcal{U}_K$ there exists $\lambda_x \neq 0$ such that $A^K \ell(x) = \lambda_x \ell(x)$. Now let us fix an element $x \in \mathcal{U}_K$ and $r > 0$ such that the open ball $\mathcal{B}$ of center x and radius r lies completely in $\mathcal{U}_K$. Consider $y \in \mathcal{B}$ and $z = (x + y)/2$. Clearly also $z \in \mathcal{B}$. A direct calculation shows that $\ell(z) = (\ell(x) + \ell(y))/2$ and then

$$0 = 2\lambda_z \ell(z) - 2A^K \ell(z) = \lambda_z(\ell(x) + \ell(y)) - (A^K \ell(x) + A^K \ell(y))$$
$$= (\lambda_z - \lambda_x)\ell(x) + (\lambda_z - \lambda_y)\ell(y).$$

Therefore, either $\lambda_x = \lambda_z = \lambda_y$ or $\ell(x)$ and $\ell(y)$ are linearly dependent eigenvectors of A^K and, hence, are associated to the same eigenvalue $\lambda = \lambda_x = \lambda_y$. This shows that $A^K \ell = \lambda \ell$ on $\mathcal{B}$. Now, let $\{\mathbf{e}_i : 1 \leq i \leq n\}$ be an arbitrary basis of $\mathbb{K}^n$ not containing x and $s > 0$ such that $x + s\mathbf{e}_i \in \mathcal{B}$ for all $i \leq n$. The set $\{\ell(x + s\mathbf{e}_i) : 1 \leq i \leq n\} \cup \{\ell(x)\}$ turns to be a basis of the whole $\mathbb{K}^{n+1}$ composed of eigenvectors of A^K with eigenvalue λ, which implies that $A^K = \lambda I$ as claimed. □

Remark 3.1. Some interesting consequences of the theorem above are:

1. From Theorem 3.1, it follows that a point $x \in \mathbb{K}^n$ is periodic with minimal period K if and only if $\ell(x)$ is an eigenvector of A^K associated to a nonzero eigenvalue, but is not an eigenvector of A^m for any $m < K$. Thus equation (1.5) has minimal K-periodic points if and only if A^K has eigenvectors which are not eigenvectors of A^m for $m = 1, 2, \ldots, K - 1$. This is only possible if A has a pair of eigenvalues λ, μ such that $\lambda^K = \mu^K$ and $\lambda^m \neq \mu^m$ for $m < K$. In addition, the corresponding eigenspace associated to λ^K must have nonempty intersection with the hyperplane $x_{n+1} = 1$ in $\mathbb{K}^{n+1}$. Some consequences can be derived from this fact. For example, if all the eigenvalues of A have different modulii, then equation (1.5) does not have any periodic orbit different from the fixed points of F. IfA is real and symmetric, then equation (1.5) has only periodic orbits of minimal periods 1 and 2. The latest is only possible if $(\lambda, -\lambda)$ is a pair of eigenvalues of A for some $\lambda \neq 0$.
2. If $n = 1$, then F is the Möbius transformation (1.2), and the matrix A has size 2×2. A direct consequence of our results gives Theorem 10.6.1 in Ref. [6]: namely, if there exists a point x such that $F^K(x) = x$, then either $F(x) = x$ or F is globally K-periodic.
3. If F is linear, that is $c_i = b_i = 0$ for $i = 1, 2, \ldots, n$, $d = 1$, then Theorem 3.1 (b) reduces to the well-known result that a linear n-dimensional map $F(x) = Mx$ is K-periodic if and only if $M^K = I$.

3.2 The case of rational coefficients

Theorem 3.1 gives us the possibility to construct globally periodic rational maps in $\mathbb{R}^n$ of the form (1.5) with any prescribed minimal period $K \geq 1$. However, in realistic uses, one works with rational coefficients. In the spirit of Ref. [6], we investigate how many minimal periods are possible with this restriction. First, we recall that, for a positive integer K, Euler's phi-function $\phi(K)$ is defined as the number of positive integers $m \geq K$ that are coprime with K (i.e. $gcd\{m, K\} = 1$). It is well known that $\phi(K)$ is the degree of the cyclotomic polynomial $\Phi_K(x)$ whose roots are exactly the primitive Kth roots of the unity. Moreover, $\Phi_K(x)$ has integer coefficients and it is irreducible over the rationals; that is, no proper factor of $\Phi_K(x)$

has rational coefficients. The following formulas are useful for the computation of $\phi(K)$ (see, e.g. [12,13]):

(i) $\phi(mn) = \phi(m)\phi(n)$ for all relatively prime integers m, n.
(ii) $\phi(p^n) = p^n - p^{n-1}$, for any prime number $p > 1$, and all integer $n \geq 1$.

As a consequence, if $K = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$ with $p_1, \dots, p_r$ different primes, then

$$\phi(K) = \phi(p_1^{n_1})\phi(p_2^{n_2})\dots\phi(p_r^{n_r}) = p_1^{n_1-1}p_2^{n_2-1}\dots p_r^{n_r-1}(p_1-1)\dots(p_r-1).$$

Proposition 3.2. *Let K and n be positive integers. Then, the following conditions are equivalent:*

(a) There exists a matrix $A \in \mathcal{M}_{n\times n}(Q)$ such that $A^K = I$ and $A^m \neq I$ for $m = 1, \dots, K-1$.
(b) $n \geq g(K)$, where

$$g(K) := \min\left\{\sum_{i=1}^{r} \phi(\mathrm{d}_i) : K = \mathrm{l.c.m.}(\mathrm{d}_1, \dots, \mathrm{d}_r)\right\}, \tag{3.2}$$

and l.c.m.$(d_1, \dots, d_r)$ means the least common multiple of $d_1, \dots, d_r$.

Proof. First we show that (a) implies (b). We will make use of the irreducible factorization of $x^K - 1$ in $\mathbb{Q}[x]$, given by the formula [12,13]

$$x^K - 1 = \prod_{d|K} \Phi_d(x), \tag{3.3}$$

where $d|K$ means that d is a positive integer divisor of K (including 1 and K).

Let $A \in \mathcal{M}_{n\times n}(Q)$ satisfying (a). Denote by $m_A(x)$ the minimal polynomial of A, and by $q_A(x)$ its characteristic polynomial. Since $q_A(x)$ and $m_A(x)$ have the same irreducible factors and $m_A(x)$ divides $x^K - 1$, it follows that the irreducible factors of $q_A(x)$ are powers of the cyclotomic polynomials $\Phi_d(x)$, $d|K$. Let m be the least common multiple of $\{d : \Phi_d(x)|q_A(x)\}$. Then, all roots of $q_A(x)$ are mth roots of the unity, and, therefore, $A^m = I$. Hence, $m = K$, and thus the degree of $q_A(x)$ is at least $g(K)$.

The proof that (b) implies (a) is constructive. First, notice that if $A_n \in \mathcal{M}_{n\times n}(Q)$ satisfies $A_n^K = I$ and $A_n^m \neq I$ for $m = 1, \dots, K-1$, then, for any $N > n$, we can construct an $N \times N$ block matrix A_N with the same property: namely,

$$A_N = \left(\begin{array}{c|c} A_n & 0 \\ \hline 0 & I \end{array}\right),$$

where I is the identity matrix of size $N - n$. Thus, it is enough to find the matrix A for $n = g(K)$. If $d_1, \dots, d_r$ satisfy $K = $ l.c.m.$(d_1, \dots, d_r)$ and $g(K) = \sum_{i=1}^{r} \phi(d_i)$, then we can choose $A = C\left[\prod_{i=1}^{r} \Phi_{d_i}\right]$, where $C[p]$ stands for the companion matrix to the polynomial $p(x)$ (see, e.g. [9, p. 146]). □

The following result provides an explicit formula to compute $g(K)$.

PROPOSITION 3.3. *If $K = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$ is the factorization of K into distinct prime powers, then*

$$g(K) = \begin{cases} \sum_{i=1}^{s} \phi(p_i^{n_i}) - 1 & \text{if } K > 2 \text{ is even and } 4 \text{ does not divide } K, \\ \sum_{i=1}^{s} \phi(p_i^{n_i}) & \text{otherwise.} \end{cases}$$

Proof. In the latest case, the least common multiple in equation (3.2) is attached for $d_1 = p_1^{n_1}, d_2 = p_2^{n_2}, \dots, d_s = p_s^{n_s}$ $(r = s)$. In the first one, we can assume $p_1^{n_1} = 2$. Taking into account that $\phi(2p^n) = \phi(p^n)$ if $p > 2$ is prime, we get the minimum in equation (3.2) for $r = s - 1$, and $d_1 = 2p_2^{n_2}, d_2 = p_3^{n_3}, \dots, d_{s-1} = p_s^{n_s}$. □

Remark 3.2. We point out some useful comments on $g(K)$. First, if $K > 2$ is even and $K/2$ is odd (the first case in the definition of g), then $g(K) = g(K/2)$. Notice also that, by the definition, $g(K) \leq \phi(K) < K$ holds for all positive integer K. Moreover, $g(K) = \phi(K)$ if and only if K is a prime power, $K = 2p^n$ with $p > 2$ prime, or $K = 12$.

Another interesting property is that $g(K) \leq K/2$ if K is even. Moreover, $g(K) = K/2$ if and only if K is a power of 2.

Finally, another consequence of the definition (3.2) is that if K_1, K_2 are two positive integers, then $g(\text{l.c.m.}(K_1, K_2)) \leq g(K_1) + g(K_2)$.

Proposition 3.2 says that there exists a globally periodic linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with rational coefficients and minimal period K if and only if $n \geq g(K)$. On the other hand, for any fixed positive integer n, the set of values K for which $g(K) \leq n$ is finite. Thus, the set of possible minimal periods for globally periodic linear maps with rational coefficients in $\mathbb{R}^n$ is finite. Moreover, Proposition 3.2 gives not only the way to find such a set, but also the way to construct examples of globally periodic maps with any possible minimal period using the corresponding companion matrices.

Example 3.1 A globally periodic linear map in $\mathbb{R}^2$ or $\mathbb{R}^3$ with rational coefficients may only have minimal period $K \in \{1, 2, 3, 4, 6\}$.

It is easy to verify that $g(K) \leq 3 \Leftrightarrow \phi(K) \leq 3$, and $g(K) \leq 2 \Leftrightarrow \phi(K) \leq 2$. Since $\phi(K) \geq \sqrt{K}$, for all $K \neq 2, 6$ ([13]), we get $\phi(K) > 3$ for all $K \geq 10$. For $K < 10$ we compute directly $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = \phi(6) = 2$, $\phi(5) = \phi(8) = 4$, and $\phi(7) = \phi(9) = 6$. Hence, $\phi(K) \leq 3 \Leftrightarrow \phi(K) \leq 2 \Leftrightarrow K \in \{1, 2, 3, 4, 6\}$.

Using the companion matrices associated to different cyclotomic polynomials, we can easily construct globally periodic linear maps in $\mathbb{R}^2$ and $\mathbb{R}^3$ with integer coefficients and minimal periods $1, 2, 3, 4, 6$. For example, in $\mathbb{R}^2$ we have $L_1(x, y) = (x, y)$, $L_2(x, y) = (-x, y)$, $L_3(x, y) = (y, -x - y)$, $L_4(x, y) = (y, -x)$, and $L_6(x, y) = (y, -x + y)$.

Next we try to extend the result of Proposition 3.2 in order to get similar consequences for rational maps of the form (1.5). Our main result states that the admissible periods for such a rational map form a finite set; explicitly:

PROPOSITION 3.4. *For each $n \in \mathbb{N}$, let us consider $M_n = \max\{m \in \mathbb{N} : g(m) < n\}$.*

If $A \in \mathcal{M}_{n\times n}(\mathbb{Q})$ satisfies $A^K = qI$ for some $q \neq 0$ and $A^m \neq \lambda I$ for every $\lambda \neq 0$ and $m = 1, 2, \dots, K - 1$, then $K \leq nM_n$.

As a consequence, if a rational map given by equation (1.5) *on* $\mathcal{U} \subset \mathbb{R}^n$ *with rational coefficients is globally periodic with minimal period K, then* $K \leq (n+1)M_{n+1}$.

Proof. Let $\lambda \in \mathbb{C}$ be any of the characteristic roots of A and let $s \in \mathbb{N}$ be the smallest integer such that $|\lambda|^s \in \mathbb{Q}$. Now, let us consider $r \in \mathbb{N}$ such that $|\lambda|^r \in \mathbb{Q}$; then $s \leq r$ and, thus, there exists $j \in \mathbb{N}$ such that $js \leq r < (j+1)r$. If $js \neq r$ then $|\lambda|^{r-js} \in \mathbb{Q}$ and hence $r - js \geq s$, a contradiction, which shows that $js = r$. One then obviously gets that s divides both n and K since $|\lambda|^n = |\det(A)| \in \mathbb{Q}$ and $|\lambda|^K = |q| \in \mathbb{Q}$. Let us denote $p = K/s$. We have that $(|1/\lambda|^s A^s)$ is a root of the identity with minimal period p and, according to Proposition 3.2, it follows that $n \geq g(p)$. This clearly means that $p \leq M_n$ and, consequently, $K = sp \leq sM_n \leq nM_n$, as claimed.

The second part of the statement is now straightforward bearing in mind Theorem 3.1. □

Proposition 3.4 gives an upper bound for the possible minimal periods of a globally periodic rational map with rational coefficients defined by equation (1.5). However, this bound is far from being sharp, as in the case of linear maps. Our next results in this section are devoted to strengthen this result.

First, we show that the aforementioned upper bound cannot be less than $g(K)$.

Proposition 3.5. *Let K and n be positive integers. If* $n \geq g(K)$, *then there exists a matrix* $A \in \mathcal{M}_{n\times n}(\mathbb{Q})$ *such that* $A^K = qI$ *for some* $q \neq 0$ *and* $A^m \neq \lambda I$ *for every* $\lambda \neq 0$ *and* $m = 1, 2, \ldots, K-1$.

Proof. For $n = 1$, it is trivial, so we set $n \geq 2$. As in the proof of Proposition 3.2, it is enough to consider the case $n = g(K)$. Moreover, in general, we can choose the same matrices A such that $A^K = I$. The unique exception are those cases such that K is even, $A^K = I$ and $A^{K/2} = -I$. To solve this, notice that $A^{K/2} = -I$ implies that $m_A(x) | (x^{K/2} + 1)$, and hence $m_A(x) = (x^{K/2} + 1)$ (since $(x^{K/2} + 1)$ is irreducible). Thus, $g(K) = n \geq K/2$. In view of Remark 3.2, this is only possible if $K = 2^m$ for some positive integer m. In this case, $n = g(K) = \phi(K) = K/2$ and $C[\Phi_K]^K = I$, $C[\Phi_K]^n = -I$. Since $n = g(K) = g(2^m) = \phi(2^m) = \phi(3.2^{m-1}) = \phi(3K/2)$, we can choose $A = C[\Phi_{3K/2}]$. Then $A \in \mathcal{M}_{n\times n}(\mathbb{Q})$, $A^K = I$, $A^m \neq \pm I$ for $m = 1, \ldots, K-1$. We are done. □

Our next result is a partial converse of Proposition 3.5. In the sequel, we will denote by Ker(A) and Im(A) respectively the null space and the range of a matrix A.

Proposition 3.6. *Let K and n be positive integers. Suppose that there exists a matrix* $A \in \mathcal{M}_{n\times n}(\mathbb{Q})$ *such that* $A^K = qI$ *for some* $q \neq 0$ *and* $A^m \neq \lambda I$ *for every* $\lambda \neq 0$ *and* $m = 1, 2, \ldots, K-1$. *Then* $n \geq g(K)$ *holds in either of the following cases:*

(i) *K and n are coprime*.
(ii) *K is prime*.
(iii) *n is prime*.

Proof.

(i) If $s \in \mathbb{N}$ is the smallest integer such that $|\lambda|^s \in \mathbb{Q}$ for a characteristic root $\lambda \in \mathbb{C}$ of A, one sees as in the proof of Proposition 3.4 that s must divide both K and n but, since K and n are coprime, this is only possible if $s = 1$ and therefore $|\lambda| \in \mathbb{Q}$. Now if $q > 0$, then the matrix $B = (1/|\lambda|)A$ satisfies $B^K = I$ with minimal period K which by Proposition 3.2 leads to $n \geq g(K)$. If, instead, $q < 0$, then $B^{2K} = 1$ and it can be easily seen that $2K$ is the smallest integer for which this occurs; again, by Proposition 3.2, we have $n \geq g(2K) \geq g(K)$.

(ii) In view of (I), we can assume that K and n are not coprime. Since K is prime, this means that $n \geq K > g(K)$.

(iii) For $n = 2$, this result was obtained in [6, Theorem 10.6.2]. Hence, we may suppose that n is odd. Further, if $K \leq n$ then the result follows from $g(K) \leq \phi(K) < K$. Let us then consider $K > n$. Since n is odd, A admits a real eigenvalue λ_1. Clearly, $\lambda_1^n = \pm\det(A) \in \mathbb{Q}$. The non-null, A-invariant subspace $V = \operatorname{Ker}(A^n - \lambda_1^n I)$ admits a basis $\mathcal{B} \subset \mathbb{Q}^n$. Let us consider the associated matrix M of the restriction $A_{|V}$ with respect to such basis. Obviously, $M \in \mathcal{M}_{r\times r}(\mathbb{Q})$ where $r = \dim(V) < n$ since $K > n$ and $M^n = \lambda_1^n I$. Therefore, the minimal polynomial of M must (strictly) divide $x^n - \lambda_1^n$. This means that $x^n - \lambda_1^n$ is reducible in $\mathbb{Q}[x]$. Since n is prime, in view of Theorem 9.1 in Ref. [12], this is only possible if $\lambda_1^n = \alpha^n$ for some $\alpha \in \mathbb{Q}$. Now, the injectivity of the mapping $f(x) = x^n$ on $\mathbb{R}$ for odd n shows that $\lambda_1 = \alpha \in \mathbb{Q}$. Since $\alpha^K = \lambda_1^K = q$, we have that $(\alpha^{-1}A)^K = I$ and $(\alpha^{-1}A)^m \neq I$ for $0 < m < K$, which according to Proposition 3.2, implies $n \geq g(K)$. □

Remark 3.3. One might think that the condition $|\lambda| \in \mathbb{Q}$ is always satisfied, but this is not the case even for $K > n$. For instance, the companion matrix $C[p]$ of the polynomial $p(x) = x^6 + 3x^3 + 9$ has all its characteristic roots of modulus $\sqrt[3]{3}$ and $C[p]^9 = 27I$. Actually this is also a possibility in the case that n is prime and $K \leq n$; for example, if $q(x) = x^3 - 3$ then $C[q]^3 = 3I$.

In the light of Propositions 3.5 and 3.6, we guess that the following statement is true, although we are not able to prove it at this moment.

CONJECTURE 3.1. *Let K and n be positive integers. Then, the following conditions are equivalent:*

(a) There exists a matrix $A \in \mathcal{M}_{n\times n}(\mathbb{Q})$ such that $A^K = qI$ for some $q \neq 0$ and $A^m \neq \lambda I$ for every $\lambda \neq 0$ and $m = 1, 2, \ldots, K-1$.

(b) $n \geq g(K)$.

For rational maps of the form (1.5), the following result follows easily from Theorem 3.1 and Propositions 3.5 and 3.6.

COROLLARY 3.7. *Let $F : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{U}$ be a globally periodic map defined by equation* (1.5) *with $a_{ij}, b_i, c_i, d \in \mathbb{Q}$.*

(a) *If $n+1$ is prime, then F may have minimal period K if and only if $g(K) \leq n+1$.*
(b) *If K is prime, then F may have minimal period K if and only if $n \geq K-2$.*

Remark 3.4. In view of Example 3.1, Corollary 3.7 indicates that, although one can find globally periodic maps in $\mathbb{R}^2$ with rational coefficients of the form (1.5) which are not linear, no new minimal period appears. The situation in $\mathbb{R}^3$ is different, since we can easily construct globally periodic rational functions in $\mathbb{R}^3$ of the form (1.5) with rational coefficients and minimal period $K \in \{1, 2, 3, 4, 6\} \cup \{5, 8, 10, 12\}$. This latest set is obtained as the set of numbers K for which $g(K) = 4$.

Example 3.2. The map

$$F(x, y, z) = \frac{-1}{x+y+z+1}(y, z, 1)$$

is well defined in $\mathcal{U} = \{(x, y, z) \in \mathbb{R}^3 : xyz(x+y+z+1) \neq 0\}$, and it is globally 5-periodic in $\mathcal{U}$. Moreover, all points in $\mathcal{U}$ have minimal period 5.

Open Problem. We notice that there exist rational maps in $\mathbb{R}^2$ with integer coefficients and minimal period different from $1, 2, 3, 4, 6$. For example, the Lyness recurrence (see, e.g. Ref. [5] and references therein) provides the map $F(x, y) = (y, (1+y)/x)$, which has minimal period 5. It would be interesting to find the set of all possible minimal periods of globally periodic rational maps in $\mathbb{R}^2$ with rational coefficients. According the results in Ref. [14], if F comes from a difference equation

$$x_{m+2} = \frac{a_0 + a_1 x_m + a_2 x_{m+1}}{b_0 + b_1 x_m + b_2 x_{m+1}}, \quad m = 0, 1, \ldots,$$

where $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{Q}$, then the unique possible minimal periods are $1, 2, 3, 4, 5, 6, 8, 12$. In fact, periods $1, 2, 3, 4, 6$ and their respective double values $2, 4, 6, 8, 12$ appear, respectively, in the Möbius generated transformations of first-order ($a_1 = b_1 = 0$) and second-order ($a_2 = b_2 = 0$). Finally, period 5 recurrences are only those equivalent to the Lyness one ($a_0 = a_2 = b_1 = 1$, $a_1 = b_0 = b_2 = 0$).

4. Eventual global periodicity of *F*

In this section, we address the problem of eventual global periodicity in maps F defined by equation (1.5).

Let $\mathcal{U}$ be an open subset of $\mathbb{K}^n$ and consider a map $F : \mathcal{U} \to \mathcal{U}$. We will say that point $x \in \mathcal{U}$ is eventually K-periodic if it is not K-periodic but there exists $m \in \mathbb{N}$ such that $F^m(x)$ is K-periodic.

The map F is said to be *globally eventually periodic* whenever F is not globally K-periodic but $F^{K+m} = F^m$ for some $m \in \mathbb{N}\backslash\{0\}$.

Theorem 4.1. *Let $F(x) = (F_1(x), \ldots, F_n(x))$ be defined by equation* (1.5) *and let $A \in \mathcal{M}_{n+1 \times n+1}(K)$ be a matrix such that $F = qA\ell$.*

(a) *A point $x \in \mathbb{K}^n$ is eventually K-periodic if and only if $\ell(x) = v + u$ where v is an eigenvector of A^K associated to a non-zero eigenvalue and $0 \neq u \in \mathrm{Ker}(A^m)$ for some $m \geq 1$.*

(b) *F is globally eventually K-periodic if and only if A is singular and the restriction $\tilde{A}$ of A to $\mathrm{Im}(A^{n+1})$ satisfies $\tilde{A}^K = \lambda \mathrm{Id}$ for some $\lambda \neq 0$, where Id stands for the identity map of $\mathrm{Im}(A^{n+1})$.*

Proof.

(a) Let us consider the well-known decomposition $\mathbb{K}^{n+1} = \mathrm{Im}(A^{n+1}) \bigoplus \mathrm{Ker}(A^{n+1})$. Recall that both $\mathrm{Im}(A^{n+1})$ and $\mathrm{Ker}(A^{n+1})$ are invariant under A and that the restrictions of A to $\mathrm{Im}(A^{n+1})$ and $\mathrm{Ker}(A^{n+1})$ are respectively invertible and nilpotent. Suppose that $x \in \mathbb{K}^n$ is eventually K-periodic with $F^K(F^m(x)) = F^m(x)$ for some $m \in \mathbb{N}\backslash\{0\}$. By Proposition 2.3 and Lemma 2.1, we have that $A^K A^m \ell(x) = \lambda A^m \ell(x)$ for some $\lambda \neq 0$. According to the decomposition above, there exist $v \in \mathrm{Im}(A^{n+1})$ and $u \in \mathrm{Ker}(A^{n+1})$ such that $\ell(x) = v + u$ and we get $A^{K+m}v + A^{K+m}u = \lambda A^m v + \lambda A^m u$. Since $\mathrm{Im}(A^{n+1})$ and $\mathrm{Ker}(A^{n+1})$ are invariant under A, this last equation implies $A^{K+m}v = \lambda A^m(v)$ and $A^K A^m u = \lambda A^m u$. Now, $\lambda \neq 0$ and the fact that the restriction of A to $\mathrm{Ker}(A^{n+1})$ is nilpotent clearly imply that $A^m u = 0$. Further, if we denote by T the inverse of the restriction of A to $\mathrm{Im}(A^{n+1})$ then we have that $A^K v = T^m A^{K+m} v = T^m(\lambda A^m v) = \lambda v$, which shows that v is an eigenvector of A^K. Finally, $u \neq 0$ for otherwise, according to Theorem 3.1, x would be K-periodic.

Conversely, if $\ell(x) = v + u$ where $A^K v = \lambda v$ with $\lambda \neq 0$ and $u \in \mathrm{Ker}(A^m)$ for $m \geq 1$, then $F^{K+m}(x) = q(A^{K+m}v + A^{K+m}u) = q(\lambda A^m v) = q(A^m v) = q(A^m(v+u)) = F^m(x)$, which shows that x is eventually K-periodic.

(b) Suppose that $\tilde{A}^K = \lambda \mathrm{Id}$ where $\lambda \neq 0$ and let $\nu \in \mathbb{N}\backslash\{0\}$ be the index of A (this is to say, ν is the smallest positive integer such that $\mathrm{Ker}(A^{n+1}) = \mathrm{Ker}(A^\nu)$). It is clear that $\mathrm{Im}(A^{n+1}) = \mathrm{Im}(A^\nu)$ and hence $\tilde{A}$ is the restriction of A to $\mathrm{Im}(A^\nu)$. Take $x \in \mathcal{U}_{K+\nu}$ and let $v \in \mathrm{Im}(A^\nu)$, $u \in \mathrm{Ker}(A^\nu)$ be such that $\ell(x) = v + u$. We then have that $F^{K+\nu}(x) = q(A^{K+\nu}v + A^{K+\nu}u) = q(\lambda A^\nu v) = q(A^\nu v) = q(A^\nu(v+u)) = F^\nu(x)$, showing that F is eventually K-periodic.

For the converse, suppose that there exists $m \in \mathbb{N}$, $m \geq 1$ such that $F^{K+m} = F^m$ on $\mathcal{U}_{K+m}$. By the first part, for all $x \in \mathcal{U}_{K+m}$ one has $\ell(x) = v_x + u_x$ where $A^K v_x = \lambda_x v_x$ and $u_x \in \mathrm{Ker}(A^m)$. Let us fix a point $x \in \mathcal{U}_{K+m}$ and consider again $r > 0$ such that the open ball $\mathcal{B}$ of center x and radius r lies completely on $\mathcal{U}_{K+m}$. If $y \in \mathcal{B}$, $z = (x+y)/2$ and $\ell(y) = v_y + u_y$, $\ell(z) = v_z + u_z$ are the corresponding decompositions with $A^K v_y = \lambda_y v_y$, $A^K v_z = \lambda_z v_z$ and $u_y, u_y \in \mathrm{Ker}(A^m)$, then from $2\ell(z) = \ell(x) + \ell(y)$ one immediately gets that $2v_z = v_x + v_y$ and $2u_z = u_x + u_y$. As in the proof of Theorem 3.1, one verifies that $\lambda_x = \lambda_y$. Let us denote $\lambda = \lambda_x$. Again, for a basis $\{\mathbf{e}_i : 1 \leq i \leq n\}$ of $\mathbb{K}^n$ not containing x and $s > 0$ such that $x + s\mathbf{e}_i \in \mathcal{B}$ for all $i \leq n$, one gets that $\{\ell(x + r\mathbf{e}_i) : 1 \leq i \leq n\} \cup \{\ell(x)\}$ linearly spans $\mathbb{K}^{n+1}$. Thus, if we denote $v_0 = v_x, u_0 = u_x$ and for each $i \leq n$ we set $\ell(x + r\mathbf{e}_i) = v_i + u_i$, where v_i is a λ-eigenvector of A^K and $u_i \in \mathrm{Ker}(A^m)$, then every $w \in \mathbb{K}^{n+1}$ may be written $w = \sum_{i=0}^n \alpha_i(v_i + u_i)$. This clearly implies that $A^{n+1}w = \sum_{i=0}^{n+1} \alpha_i A^{n+1} v_i$ since $\mathrm{Ker}(A^m) \subseteq \mathrm{Ker}(A^{n+1})$ and hence

$$A^K A^{n+1} w = \sum_{i=0}^{n+1} \alpha_i A^{n+1} A^K v_i = \lambda \sum_{i=0}^{n+1} \alpha_i A^{n+1} v_i = \lambda A^{n+1} w,$$

which proves that A^K is a multiple of the identity on Im(A^{n+1}). Finally, if A was invertible, then F would be K-periodic according to Theorem 3.1; therefore A must be singular in order to exclude this case. □

Remark 4.1. The results of section 3.2 together with the theorem above show that if a map F defined by equation (1.5) on an open set $\mathcal{U} \subset \mathbb{K}^n$ has all its coefficients rational, then global eventual K-periodicity of F implies that $K < nM_n$, where M_n is defined as in Proposition 3.4. Further, for a prime integer K it is easy to see that such a globally eventually periodic map F may have minimal period K if and only if $n \geq K - 1$. Both results follow immediately bearing in mind that a corresponding matrix must be singular.

Acknowledgements

The authors thank Professors Armengol Gasull and Javier Majadas for useful discussions, and the referee for his/her interesting remarks. This work has been partially supported by M.E.C. (Spain) and FEDER, under grant MTM2004-06652-C03-02.

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