IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **27** (2014) 289–304 doi:10.1088/0951-7715/27/2/289

Global dynamics of discrete neural networks allowing non-monotonic activation functions

Eduardo Liz[1] **and Alfonso Ruiz-Herrera**[2,3]

[1] Departamento de Matemática Aplicada II, Universidad de Vigo, 36310 Vigo, Spain
[2] Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

E-mail: eliz@dma.uvigo.es and alfonsoruiz@ugr.es

Received 25 April 2013, revised 9 December 2013
Accepted for publication 30 December 2013
Published 17 January 2014

Recommended by L Bunimovich

Abstract
We show that in discrete models of Hopfield type some properties of global stability and chaotic behaviour are coded in the dynamics of a related one-dimensional equation. Using this fact, we obtain some new results on stability and chaos for a system of delayed neural networks; some relevant properties of our results are that we do not require monotonicity properties in the activation function, we allow any architecture of the network, and the conclusions are independent of the size of the time delays.

Keywords: neural networks, global stability, chaotic dynamics

Mathematics Subject Classification: 39A30, 39A33, 37D45, 92B20

(Some figures may appear in colour only in the online journal)

1. Introduction

Hopfield proposed a neural network model where each neuron is considered as a linear circuit formed by a resistor and a capacitor [16, 17]. This model is governed by a system of ordinary differential equations where neurons are linked via an activation function and the updating and communication in the network is instantaneous. Later, Marcus and Westervelt [31] introduced time delays in Hopfield's model, accounting for the fact that the propagation velocity of neural

[3] Present address: Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary.

signals or the switching speed of amplifiers (neurons) is finite. Nowadays, time delays are omnipresent in most of models in neural networks.

In this paper we consider the discrete model of Hopfield type given by the n equations

$$x_i(N+1) = \alpha x_i(N) + \sum_{j=1}^{n} b_{ij} g(x_j(N - k_{ij})), \ N = 0, 1, \dots . \tag{1.1}$$

System (1.1) describes the evolution of n-neurons when the state of the network is updated discretely, see, e.g., [19, 20, 22, 23, 33, 37, 42, 45, 46, 50]; $x_i(N)$ represents the voltage of neuron i at the time N, $\alpha \in (0, 1)$ is the internal decay of the neurons, and the constant b_{ij} is the connection weight between neuron i and neuron j. If the output from neuron j excites (respectively, inhibits) neuron i, then $b_{ij} > 0$ (respectively, $b_{ij} < 0$). We shall assume that the matrix $B = (b_{ij})$ is irreducible [18], which means that the considered network is strongly connected. Constants $k_{ij} \in \mathbb{N}$ represent the time delays. Finally, the map $g : \mathbb{R} \longrightarrow \mathbb{R}$ is the activation function of the network. In most of the related literature, the activation function is sigmoid, that is

- g is of class $\mathcal{C}^1$, $g'(x) > 0$ for all $x \in \mathbb{R}$, and $g'(0) = \sup\{g'(x) : x \in \mathbb{R}\}$,
- $g(0) = 0$,
- g is bounded.

For some practical purposes, it is convenient to consider non-monotonic activation functions in system (1.1), for instance, to increase the performance of the associative memory [34, 35]. However, as indicated in [29], a full mathematical analysis of such networks may still be remote, especially when time delays are considered. In this paper, we allow the activation functions to be quite general, including in particular the prototype of non-monotonic output introduced by Morita [34].

The first part of our analysis, developed in section 2, focuses on the problem of global stability and multistability for system (1.1). In this regard, our main result can be summarized as follows: assume that all connections are excitatory (i.e. $b_{ij} \geqslant 0$) and the connection matrix is normalized as

$$\sum_{j=1}^{n} b_{ij} = L > 0$$

for all $i = 1, \dots, n$ (this condition was originally introduced in [31]). In this framework, some properties of global attraction in system (1.1) are coded in the one-dimensional equation

$$x(N+1) = f(x(N)), \ N = 0, 1, \dots, \tag{1.2}$$

where $f(x) = (L/(1-\alpha))g(x)$. Specifically, a global attractor or multistability in (1.1) produces a global attractor or multistability in (1.2). It is worth noting that our results do not require the usual hypotheses of monotonicity or contractivity (see, e.g., [12, 28, 33, 41] and references therein) and are independent of the size of the delay. Moreover, in some architectures, such as ring networks with an even number of neurons [2], our results are optimal; see theorem 2.3. In real situations, determining the conditions under which an equilibrium is globally asymptotically stable regardless the size of the time delays is especially important in the practical design of artificial neural networks [44]. On the other hand, the co-existence of several stable equilibria in a neural network has an outstanding effect in pattern recognition and associative memory, see [7, 13, 15, 17].

In the second part of our paper (section 3), we address the issue of chaos in system (1.1). Although numerous experiments have demonstrated the presence of irregular patterns in neuroscience, see, for instance, [39], the question of how these behaviours appear in neural networks models is not completely understood from a theoretical point of view. Using the approach introduced in [26], we prove the existence of chaos in system (1.1) based on some elementary notions of chaotic dynamics in one-dimension. In our results about chaos, we do not impose excitation connections, nor the previous normalization, nor small/large parameters. An important aim of these results is to check analytically some classical properties of complex dynamics in system (1.1). For instance, we estimate the sensitive dependence on the initial conditions and explicit regions in the phase space with chaotic behaviour. As explained in [19], chaotic neural networks have potential applications in practical problems.

We emphasize that, for some activation functions, stable equilibria with large basins of attraction may exist in some parameter ranges where the system displays chaotic behaviour. This aspect, together with other implications of our results, is presented in section 4. Finally, we point out that the idea of relating the dynamics of a one-dimensional map to the dynamics of a more complicated system has been successfully employed not only in difference equations [10,11,24], but also in delay-differential equations (see, e.g., [21,25,30] and references therein), and partial differential equations [47,48]. As far as we know, this is the first time that this idea is applied to systems of discrete neural networks; for an application to the continuous case, see the recent paper [27].

2. Global stability in system (1.1)

Following our discussion in section 1, after rescaling either the coefficients or the activation function, we can write system (1.1) in the form

$$x_i(N+1) = \alpha x_i(N) + (1-\alpha)\sum_{j=1}^{n} a_{ij} f(x_j(N-k_{ij})),\ N = 0, 1, \ldots, \tag{2.1}$$

for $i = 1, \ldots, n$, where $k_{ij} \in \mathbb{N}$, $a_{ij} \in \mathbb{R}$, for all $i, j \in \{1, 2, \ldots, n\}$.

In this section we assume the following hypotheses:

(A1) $\alpha \in (0, 1)$,
(A2) if $n \geqslant 2$, then the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is irreducible and all its entries are nonnegative,
(A3) for all $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} a_{ij} = 1.$$

For the activation function $f : \mathbb{R} \longrightarrow \mathbb{R}$, we work with the following conditions:

(B1) $f(0) = 0$,
(B2) f is bounded,
(B3) $f((0, +\infty)) \subset (0, +\infty)$ and $f((-\infty, 0)) \subset (-\infty, 0)$,
(B4) f is of class $\mathcal{C}^1$ and $f'(0) > 1$.

Our first purpose is to obtain criteria of global attraction and multistability in equation (2.1) based on some dynamical properties of the one-dimensional equation

$$y(N+1) = f(y(N)),\ N = 0, 1, \ldots \tag{2.2}$$

We recall that an equilibrium p of (2.2) is a global attractor of (2.2) in a real interval J if $\lim_{N\to\infty} y(N) = p$, for all solutions $y(N)$ starting at an initial point $y(0) \in J$, that is, if $\lim_{N\to\infty} f^N(y(0)) = p$, for all $y(0) \in J$, where f^N is the Nth iteration of f under composition. A global attractor p for a one-dimensional map is always asymptotically stable (see, e.g., [10] and its references), and so it is usually referred to as a globally asymptotically stable equilibrium.

Without lost of generality, we can choose $\mathbb{R}^d$ as the phase space for equation (2.1), where

$$d = n(1 + h^*) \qquad \text{and} \qquad h^* = \max\{k_{ij} \,:\, i, j = 1, \ldots, n\}.$$

The notation employed in this section is described as follows. Given $\phi = (x_1(-h^*), \ldots, x_1(0), \ldots, x_n(-h^*), \ldots, x_n(0)) \in \mathbb{R}^d$,

$$x(\phi) = \{(x_1(N, \phi), x_2(N, \phi), \ldots, x_n(N, \phi))\}_{N \geqslant -h^*}$$

refers to the sequence obtained from (2.1) with initial condition at ϕ. The N-term of the ith component of $x(\phi)$ is denoted by $x_i(N, \phi)$, however, for simplicity, we omit the dependence on ϕ and simply write $x_i(N)$ if there is no possible confusion on the initial condition.

An equilibrium $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ of (2.1) is a global attractor of (2.1) in a set $D \subset \mathbb{R}^d$ if $\lim_{N\to\infty} x_i(N, \phi) = p_i$, for all $i = 1, 2, \ldots, n$ and every initial condition $\phi \in D$. We use the coordinate-wise partial ordering in $\mathbb{R}^d$ induced by the cone $[0, +\infty)^d$. Thus, we say that $y = (y_1, y_2, \ldots, y_d)$ is nonnegative if $y_i \geqslant 0$ for all $i = 1, 2, \ldots, d$, and y is positive if $y_i > 0$ for all $i = 1, 2, \ldots, d$. It is important to note that, under conditions (A1)–(A3), $p \in \mathbb{R}$ is an equilibrium of (2.2) if and only if $(p, \ldots, p) \in \mathbb{R}^n$ is an equilibrium of (2.1). Next we present a sharp criterion of global attraction for the null solution.

Theorem 2.1. *Assume that conditions (A1)–(A3) and (B1)–(B3) hold. Then* $0 \in \mathbb{R}$ *is a global attractor of (2.2) in* $\mathbb{R}$ *if and only if* $(0, \ldots, 0) \in \mathbb{R}^n$ *is a global attractor for (2.1) in* $\mathbb{R}^d$ *regardless of the values of the delays.*

Proof. First, we prove that given an initial condition $\phi \in \mathbb{R}^d$, the corresponding solution

$$x = x(\phi) = \{(x_1(N), x_2(N), \ldots, x_n(N))\}_{N \geqslant -h^*}$$

satisfies that

$$\lim_{N\to\infty} x_i(N) = 0,$$

for all $i = 1, \ldots, n$ provided $0 \in \mathbb{R}$ is a global attractor of (2.2) in $\mathbb{R}$. We split this proof into two steps:

Step 1. The solution x is bounded.

For the sake of contradiction, we suppose that the sequence

$$\{(x_1(N), x_2(N), \ldots, x_n(N)\}_{N \geqslant -h^*}$$

is unbounded. Assume, for instance, that there is an index $j \in \{1, 2, \ldots, n\}$ such that

$$\limsup_{N\to\infty} x_j(N) = \infty.$$

By an induction argument, we can take a partial sequence $\{\sigma(N)\}_{N\in\mathbb{N}}$ such that

$$x_j(\sigma(N) + 1) = \max\{x_j(m) : 0 \leqslant m \leqslant \sigma(N) + 1\}.$$

Thus, $\lim_{N\to\infty} x_j(\sigma(N) + 1) = \infty$ and, using that $x_j(\sigma(N) + 1) \geqslant x_j(\sigma(N))$ in the jth equation in (2.1), we get the inequality

$$x_j(\sigma(N) + 1) \leqslant \alpha x_j(\sigma(N) + 1) + (1 - \alpha) \sum_{i=1}^{n} a_{ji} f(x_i(\sigma(N) - k_{ji})), \ \forall\, N \in \mathbb{N}.$$

Therefore,

$$x_j(\sigma(N) + 1) \leqslant \sum_{i=1}^{n} a_{ji} f(x_i(\sigma(N) - k_{ji})), \ \forall\, N \in \mathbb{N}.$$

Since f is bounded, this is a contradiction and hence step 1 is proved. Note that the analogous argument works if we assume that

$$\liminf_{N\to\infty} x_j(N) = -\infty.$$

Step 2. $\lim_{N\to\infty} x_j(N) = 0$ *for all* $j = 1, 2, \ldots, n$.

By the previous step, we deduce that there are two constants β, γ such that

$$\beta < x_j(N) < \gamma$$

for all $j = 1, \ldots, n$ and $N \in \mathbb{N}$.

Consequently, there are two indices j_1, j_2 (possibly equal) so that

$$Q_1 = \liminf_{N\to\infty} x_{j_1}(N) = \min\{\liminf_{N\to\infty} x_j(N) : j = 1, \ldots, n\} \geqslant \beta,$$
$$Q_2 = \limsup_{N\to\infty} x_{j_2}(N) = \max\{\limsup_{N\to\infty} x_j(N) : j = 1, \ldots, n\} \leqslant \gamma.$$

Now we focus our attention on Q_1. We can obtain a partial sequence $\{\sigma(N)+1\}_{N\in\mathbb{N}}$ satisfying that

$$\lim_{N\to\infty} x_{j_1}(\sigma(N)+1) = Q_1\ ;\ \lim_{N\to\infty} x_{j_1}(\sigma(N)) = L_0\ ;\ \lim_{N\to\infty} x_i(\sigma(N) - k_{j_1 i}) = L_i,$$

with $L_i \in [Q_1, Q_2]$ for all $i = 0, 1, \ldots, n$. Taking limits as $N \to \infty$ in the j_1th equation in (2.1), we obtain that

$$Q_1 = \alpha L_0 + (1-\alpha)\sum_{i=1}^{n} a_{j_1 i}\, f(L_i) \geqslant \alpha Q_1 + (1-\alpha)\sum_{i=1}^{n} a_{j_1 i}\, f(L_i),$$

and therefore, by (A2) and (A3),

$$Q_1 \geqslant \sum_{i=1}^{n} a_{j_1 i}\, f(L_i) \geqslant \min\{f(x) : x \in [Q_1, Q_2]\}.$$

Reasoning in a similar way with Q_2, we arrive at the inequality

$$Q_2 \leqslant \max\{f(x) : x \in [Q_1, Q_2]\}.$$

Now it is clear that $[Q_1, Q_2] \subset f([Q_1, Q_2])$, and, by induction,

$$[Q_1, Q_2] \subset f^m([Q_1, Q_2])\ \forall\, m = 1, 2, \ldots$$

As $0 \in \mathbb{R}$ is an attractor of (2.2) in $\mathbb{R}$, we obtain that

$$\lim_{m\to\infty} f^m([Q_1, Q_2]) = \{0\},$$

and therefore $x_+ = Q_1 = Q_2 = \lim_{N\to\infty} x_j(N)$, for all $j = 1, 2, \ldots, n$.

Finally, we observe that if $0 \in \mathbb{R}$ is not a global attractor of (2.2) in $\mathbb{R}$, then we deduce from (B3) that there is a fixed point of f either in $(0, \infty)$ or in $(-\infty, 0)$ and so, an equilibrium of (2.1) different from $(0, \ldots, 0)$. Otherwise, $0 < f(x) < x$ for all $x > 0$, and $x < f(x) < 0$ for all $x < 0$, which clearly implies that 0 is a global attractor of (2.2). □

Next we give a criterion of multistability for (2.1) based on a similar dynamics in (2.2).

Theorem 2.2. *Assume that conditions (A1)–(A3) and (B1)–(B4) hold. If* $x_+ > 0$ *is a global attractor of (2.2) in* $(0, +\infty)$*, then* $(x_+, \ldots, x_+) \in \mathbb{R}^n$ *is a global attractor for (2.1) in* $(0, +\infty)^d$ *for all values of* k_{ij}, $i, j \in \{1, 2, \ldots, n\}$*. Analogously, if* $x_- < 0$ *is a global attractor of (2.2) in* $(-\infty, 0)$*, then* $(x_-, \ldots, x_-) \in \mathbb{R}^n$ *is a global attractor for (2.1) in* $(-\infty, 0)^d$ *for all values of* k_{ij}, $i, j \in \{1, 2, \ldots, n\}$.

Proof. We only prove the global attraction of $(x_+, \ldots, x_+) \in (0, \infty)^n$. The proof for $(x_-, \ldots, x_-) \in (-\infty, 0)^n$ then follows using the change of variables $y_i = -x_i$, which transforms (2.1) into the same system, but with activation function $h(x) = -f(-x)$. Note that f and h satisfy conditions (B1)–(B4) simultaneously.

Under conditions (A1), (A2) and (B3), it is clear that the solution $x(\phi)$ of (2.1) starting at a positive initial condition $\phi \in \mathbb{R}^d$ is positive, that is, $x_i(N, \phi) > 0$ for all $i = 1, 2, \ldots, n$, and all $N \in \mathbb{N}$. We shall prove that if $\phi \in \mathbb{R}^d$ is positive and

$$x = x(\phi) = \{(x_1(N), x_2(N), \ldots, x_n(N))\}_{N \geqslant -h^*}$$

is the corresponding solution, then

$$\lim_{N\to\infty} x_i(N) = x_+,$$

for all $i = 1, \ldots, n$.

First we observe that, by the proof of theorem 2.1, the solution x is bounded. Next, we prove that there exists $\delta > 0$ such that $\liminf_{N\to\infty} x_i(N) \geqslant \delta$ for all $i = 1, 2, \ldots, n$. Indeed, assume by contradiction that there are an index j and a partial sequence $\{\sigma(N)\}_{N\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} x_j(\sigma(N)) = 0.$$

Since x is positive, the expression of the jth equation of (2.1) leads to the inequality

$$x_j(\sigma(N)) \geqslant \alpha x_j(\sigma(N) - 1),$$

for all N. Thus, using that $\alpha > 0$, we obtain

$$\lim_{N\to\infty} x_j(\sigma(N) - 1) = 0,$$

and an induction argument shows that

$$\lim_{N\to\infty} x_j(N) = 0. \tag{2.3}$$

By (A2), we can take an index $r \neq j$ so that $a_{jr} > 0$. Our aim is to prove that

$$\lim_{N\to\infty} x_r(N) = 0. \tag{2.4}$$

By contradiction, assume that there are a real constant $\nu > 0$ and a partial sequence $\{\tau(N)\}_{N\in\mathbb{N}}$ such that

$$x_r(\tau(N)) > \nu > 0,$$

for all $N \in \mathbb{N}$. Now, we define

$$R := \min\{f(x) : x \in [\nu, M]\},$$

where $M > 0$ is an upper bound of $\{x_r(N)\}$, see the previous step. It follows from (B3) that $R > 0$. Using the expression of the jth equation of (2.1), we arrive at

$$x_j(\tau(N) + 1 + k_{jr}) \geqslant (1 - \alpha)a_{jr} R > 0$$

for all $N \in \mathbb{N}$. Clearly, this inequality contradicts (2.3), and thus (2.4) is proved.

Finally, repeating the previous arguments and using the irreducibility of A, we arrive at

$$\lim_{N\to\infty} x_i(N) = 0, \qquad \forall i = 1, 2, \ldots, n. \tag{2.5}$$

On the other hand, using (B1) and (B4) we can choose $\varepsilon_1, \varepsilon_2 > 0$ satisfying that

$$f(x) \geqslant (1 + \varepsilon_1)x,$$

for all $x \in [0, \varepsilon_2]$. Let N_0 be an integer such that $x_i(N) \leqslant \varepsilon_2$ for all $i = 1, 2, \ldots, n$ and $N = N_0 - d, \ldots, N_0$. Let $\tilde{x} = \min\{x_i(m) : i = 1, \ldots, n,\ m = N_0 - d, \ldots, N_0\}$. Then

$$x_i(N_0 + 1) \geqslant \alpha\tilde{x} + (1-\alpha)\sum_{j=1}^{n} a_{ij}(1+\epsilon_1)\tilde{x} > \tilde{x}, \qquad \forall\, i = 1, 2, \ldots, n.$$

This leads to a contradiction with (2.5), and the proof of this claim is complete.

To finish the proof of the theorem, we note that, using the same argument employed in step 2 of theorem 2.1, we obtain that $\lim_{N\to\infty} x_j(N) = x_+$ for all $j = 1, 2, \ldots, n$. □

Some remarks are in order. First, for some architectures, the sufficient condition for global attraction in (2.1) established in theorem 2.2 is also necessary. Indeed, let us consider the ring of neurons (see, e.g., [2])

$$\begin{aligned}
&x_1(N+1) = \alpha x_1(N) + (1-\alpha) f(x_{2n}(N - k_{1(2n)}))\\
&x_2(N+1) = \alpha x_2(N) + (1-\alpha) f(x_1(N - k_{21}))\\
&\vdots\\
&x_{2n}(N+1) = \alpha x_{2n}(N) + (1-\alpha) f(x_{2n-1}(N - k_{(2n)(2n-1)})).
\end{aligned} \tag{2.6}$$

Note that for this system, the matrix A is the cyclic permutation matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Assume (A1) and (B1)–(B4) hold. If $(x_+, \ldots, x_+) \in (0, \infty)^{2n}$ is a global attractor for (2.6) in $(0, \infty)^d$ then x_+ is a global attractor for (2.2) in $(0, \infty)$. To prove this claim we reason by contradiction. Assume that x_+ is not a global attractor for (2.2) in $(0, \infty)$. Then it is well known that there are two points (possibly equal) p_1, p_2 strictly positive and different from x_+ such that $f(p_1) = p_2$ and $f(p_2) = p_1$ (see [9]). Now we observe that $(p_1, p_2, p_1, p_2, \ldots, p_1, p_2) \in (0, \infty)^{2n}$ is an equilibrium for (2.6) different from $(x_+, \ldots, x_+) \in (0, \infty)^{2n}$, contradicting our claim. Thus we have proved the following result.

Theorem 2.3. *Assume that conditions (A1) and (B1)–(B4) hold, and let* $d = 2n\max\{1 + k_{1(2n)}, 1 + k_{(i+1)i} : i = 1, 2, \ldots, 2n-1\}$. *Then* $(x_+, \ldots, x_+) \in \mathbb{R}^{2n}$ *is a global attractor for (2.6) in* $(0, +\infty)^d$ *if and only if* $x_+ > 0$ *is a global attractor of (2.2) in* $(0, +\infty)$. *Analogously,* $(x_-, \ldots, x_-) \in \mathbb{R}^{2n}$ *is a global attractor for (2.6) in* $(-\infty, 0)^d$ *if and only if* $x_- < 0$ *is a global attractor of (2.2) in* $(-\infty, 0)$.

We cannot expect the conclusions of theorem 2.3 to remain true for an arbitrary architecture. Indeed, consider the simplest case of one neuron with a self-connection. Then equation (2.1) reduces to the so-called Clark model [8] in population dynamics

$$x(N+1) = \alpha x(N) + (1-\alpha) f(x(N-k)), \qquad N = 0, 1, \ldots, \tag{2.7}$$

where $0 < \alpha < 1$ and $k \in \mathbb{N}$. It is known that the positive equilibrium x_+ of (2.7) is globally attracting for values of α close to 1 in a wide family of maps f satisfying (B1)–(B4) even if x_+ is unstable for (2.2). See, for example, [10, 11, 24] and references therein.

A second remark is that some relevant activation functions may change signs in $(-\infty, 0)$ and $(0, \infty)$. In this case we can replace (B3) by

(B3′) there exists $a > 0$ such that $f(a) = f(-a) = 0$, $f((0,a)) \subset (0,a)$, and $f((-a,0)) \subset (-a,0)$.

Assuming this condition, we have the following analogous result to theorem 2.2.

Theorem 2.4. *Assume that conditions (A1), (A2), (A3), (B1), (B2), (B3′), and (B4) hold. If $x_+ > 0$ is a global attractor for (2.2) in $(0,a)$, then $(x_+,\ldots,x_+) \in \mathbb{R}^n$ is a global attractor for (2.1) in $(0,a)^d$ for all values of k_{ij}, $i,j \in \{1,2,\ldots,n\}$. Analogously, if $x_- > 0$ is a global attractor for (2.2) in $(-a,0)$, then $(x_-,\ldots,x_-) \in \mathbb{R}^n$ is a global attractor for (2.1) in $(-a,0)^d$ for all values of τ_{ij}, $i,j \in \{1,2,\ldots,n\}$.*

Proof. Under conditions (A1), (A2) and (B3′), the solution $x(\phi)$ of (2.1) starting at $\phi \in (0,a)^d$ satisfies that $0 < x_i(N,\phi) < a$ for all $i = 1,2,\ldots,n$, and all $N \in \mathbb{N}$. Our aim is to prove that if $\phi \in (0,a)^d$ and

$$x = x(\phi) = \{(x_1(N), x_2(N), \ldots, x_n(N)\}_{N \geqslant -h^*}$$

is the corresponding solution, then

$$\lim_{N\to\infty} x_i(N) = x_+,$$

for all $i = 1,\ldots,n$.

The proof follows exactly the same steps as in the proofs of theorems 2.1 and 2.2. The only difference is that condition (B3′) is used as in the first step of the proof of theorem 2.1 to show that there exists $B \in (0,a)$ such that

$$\limsup_{N\to\infty} x_i(N) \leqslant B.$$

The proof for x_- follows from a change of variables, as in theorem 2.2. □

Example 2.1. In many applications, the activation function f increases monotonically, is bounded and has exactly three fixed points $x_-, 0, x_+$, with $x_- < 0 < x_+$. If $f'(0) > 1$, an elementary argument shows that, in this situation, x_+ attracts all solutions of (2.2) with positive initial condition, and x_- attracts all solutions of (2.2) with negative initial condition. An application of theorem 2.2 ensures that, under conditions (A1)–(A3), $(x_+, \ldots x_+)$ attracts all solutions of (2.1) with positive initial data, and $(x_-, \ldots x_-)$ attracts all solutions of (2.1) with negative initial data. For example, the activation function [34]

$$f(x) = \frac{1 - \mathrm{e}^{-cx}}{1 + \mathrm{e}^{-cx}}$$

satisfies the above conditions if and only if $c > 2$.

The non-monotonic case is usually more difficult to treat. Next we will consider the function introduced by Morita [29, 34]

$$f(x) = \frac{1 - \mathrm{e}^{-cx}}{1 + \mathrm{e}^{-cx}} \cdot \frac{1 + \kappa\, \mathrm{e}^{c'(|x|-h)}}{1 + \mathrm{e}^{c'(|x|-h)}}, \tag{2.8}$$

where $c > 0$, $c' > 0$, $h > 0$ and $\kappa < 0$. See figure 1.

We shall prove a result of global stability for equation (2.2) with f defined by (2.8). For simplicity, we consider the case $c' = c$. We recall that the Schwarzian derivative of a C^3 map f is defined by the relation

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

whenever $f'(x) \neq 0$. Based on Singer's results [38], the Schwarzian derivative is a good tool for proving that a locally asymptotically stable equilibrium of a unimodal one-dimensional map f is globally asymptotically stable. We shall use an improvement of this classical result recently proved by El-Morshedy and Jiménez López (see [10, corollary 2.10]).

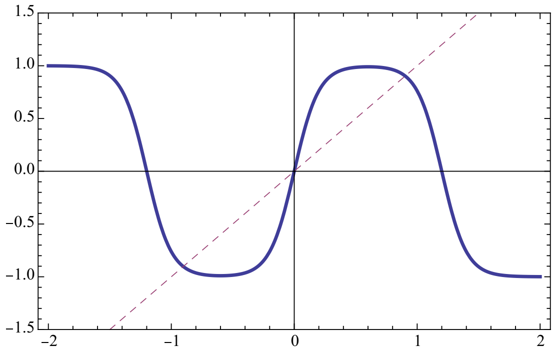

Figure 1. Graph of map f defined in (2.8) for $c = c' = 10$, $h = 1.2$, and $\kappa = -1$. The equilibria are the intersections of the solid curve $y = f(x)$ and the dashed line $y = x$.

Proposition 2.1. *Let $I = (\alpha, \beta)$ be a real interval and $f : I \to I$ a C^3 map having a unique critical point $\bar{x}$ and a unique fixed point p. Assume that one of the following conditions is satisfied.*

(i) $0 \leqslant f'(p) < 1$ *and $\bar{x}$ is a local extremum.*
(ii) $-1 \leqslant f'(p) < 0$, *$\bar{x}$ is a minimum (respectively, maximum), and $(Sf)(x) < 0$ for all $x \in (\alpha, \bar{x})$ (respectively $x \in (\bar{x}, \beta)$).*

Then p is a global attractor of f.

Next we list some properties of the map f defined in (2.8) with $c' = c$.

Lemma 2.1. *Assume that $c' = c > 0$, $\kappa < 0$, and $h > 0$ in (2.8). Then:*

(a) *f is bounded, continuous, and odd in $\mathbb{R}$, $f(x) > 0$ on $(0, a)$, and $f(x) < 0$ on (a, ∞), where $a = h - \ln(|\kappa|)/c$.*
(b) *f has three equilibria $0, x_+, x_-$, with $x_- < 0 < x_+$.*
(c) *f' is continuous and even in $\mathbb{R}$. f has two critical points $\bar{x}_+ \in (0, a)$, $\bar{x}_- = -\bar{x}_+ \in (-a, 0)$; $\bar{x}_+$ is a local maximum, and $\bar{x}_-$ is a local minimum.*
(d) *If $f(\bar{x}_+) < a$ then f maps $(0, a)$ into $(0, a)$, and $(-a, 0)$ into $(-a, 0)$.*
(e) *$f'(0) > 1$ if and only if $c > 2$ and*

$$h > \frac{1}{c} \ln\left(\frac{2 - c\kappa}{c - 2}\right). \tag{2.9}$$

(f) *f is a C^3 function in $(-\infty, 0) \cup (0, \infty)$, and*

$$(Sf)(x) < 0\,, \ \forall\, x \in ((-\infty, 0) \setminus \{\bar{x}_-\}) \cup ((0, \infty) \setminus \{\bar{x}_+\})\,.$$

Proof. Properties (a)–(e) follow from simple computations.

To prove (f), assume first that $x > 0$, $x \neq \bar{x}_+$. We write $f(x) = f_1(f_2(x))$, where

$$f_1(x) = \frac{1 - x}{1 + x} \cdot \frac{x + \kappa\, \mathrm{e}^{-ch}}{x + \mathrm{e}^{-ch}}; \qquad f_2(x) = \mathrm{e}^{-cx}.$$

Since $(Sf_2)(x) = -c^2/2 < 0$ for all $x \in \mathbb{R}$, the rule for the composition of the Schwarzian derivative (see, e.g., [38]) ensures that $(Sf)(x) < 0$ if $(Sf_1)(x) < 0$. After some computations, we can check that $(Sf_1)(x) < 0$ if and only if

$$(\kappa\, \mathrm{e}^{-ch} - 1)(\kappa - 1) > 0, \tag{2.10}$$

which trivially holds if $\kappa < 0$.

A similar proof shows that $(Sf)(x) < 0$ for all $x \in (-\infty, 0) \setminus \{\bar{x}_-\}$ if and only if (2.10) holds. □

We have the following consequence of theorem 2.4, proposition 2.1 and lemma 2.1.

Corollary 2.1. *Assume that $c' = c > 2$, $\kappa < 0$, and $h > 0$ in (2.8). Let x_-, x_+ be the negative and the positive fixed points of f, respectively, and $a = h - \ln(|\kappa|)/c$. If (2.9) holds, $f(\bar{x}_+) < a$, and $f'(x_+) \in [-1, 1)$, then $(x_+, \dots, x_+) \in \mathbb{R}^n$ is a global attractor for (2.1) in $(0, a)^d$, and $(x_-, \dots, x_-) \in \mathbb{R}^n$ is a global attractor for (2.1) in $(-a, 0)^d$.*

If $\kappa = -1$, then $a = h$, $\bar{x}_+ = h/2$ and condition $f(\bar{x}_+) < a$ holds if either $h \geqslant 1$ or $h < 1$ and $c < (2/h)\ln((1 + h^{1/2})/(1 - h^{1/2}))$.

For a numerical example, choose $c' = c = 10$, $h = 1.2$ and $\kappa = -1$ in (2.8). Then $a = h = 1.2$, $\bar{x}_+ = h/2 = 0.6$, and the positive fixed point is $x_+ \approx 0.902$. Since $h > 1$, (2.9) holds, and $f'(x_+) \approx -0.922 \in [-1, 1)$, corollary 2.1 applies to conclude that $(x_+, \dots, x_+) \in \mathbb{R}^n$ is a global attractor for (2.1) in $(0, 1.2)^d$, and $(-x_+, \dots, -x_+) \in \mathbb{R}^n$ is a global attractor for (2.1) in $(-1.2, 0)^d$. The graph of f for this particular case is represented in figure 1.

3. Chaotic dynamics in system (1.1)

In the previous section we obtained sufficient conditions for global attraction in system (2.1) from analogous properties in the one-dimensional equation (2.2). Our aim in the present section is to show that chaotic dynamics in (2.2) induces chaos in (2.1). We follow the approach developed in [26], based on the techniques from [49].

First we give the precise definition of chaotic dynamics.

Definition 3.1. *Consider (X, d) a metric space. We say that a continuous map $\psi : X \to X$ induces chaotic dynamics on two symbols if there exist two disjoint compact sets $N_0, N_1 \subset X$ such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(\omega_i)_{i\in\mathbb{Z}} \in (N_0 \cup N_1)^{\mathbb{Z}}$ such that*

$$\omega_i \in N_{s_i} \quad \text{and} \quad \omega_{i+1} = \psi(\omega_i) \quad \text{for all } i \in \mathbb{Z}, \tag{3.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i$, $\forall\, i \in \mathbb{Z}$) for some $k \geqslant 1$, there exists a k-periodic sequence $(\omega_i)_{i\in\mathbb{Z}} \in (N_0 \cup N_1)^{\mathbb{Z}}$ satisfying (3.1).

Note that if a map is chaotic according to our definition, then it is also chaotic in the sense of Block–Coppel and in the sense of coin tossing (see [1]). On the other hand, our definition of chaos guarantees natural properties of complex dynamics such as sensitive dependence on the initial conditions, the presence of an invariant set with infinitely many periodic points, and topological transitivity (see theorem 2.2 in [32]). Next we give a simple method to estimate the sensitive dependence on the initial conditions in a chaotic regime. Throughout this section, $\|\cdot\|$ denotes the usual maximum norm in $\mathbb{R}^n$. The following proposition was proved in [26].

Proposition 3.1. *Assume that $F : \mathcal{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has chaotic dynamics on two symbols relative to N_0, N_1, and denote $d = \mathrm{dist}(N_0, N_1) > 0$. For $\varepsilon > 0$, we define*

$$S_\varepsilon = \max\{n \in \mathbb{N} : N_0 \text{ contains } n \text{ disjoint balls of diameter } \varepsilon\},$$

and

$$N^* = 1 + \left\lceil \frac{\ln S_\varepsilon}{\ln 2} \right\rceil,$$

where, for $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the ceiling of x, that is, the smallest integer not less than x.

Then, there are two points x_0, y_0 so that $x_0, y_0 \in N_0$, $\|x_0 - y_0\| < \varepsilon$, and $\max_{0 \leqslant j \leqslant N^*}\{\|F^j(x_0) - F^j(y_0)\|\} > d$.

Next we introduce the following notion of strictly turbulence for one-dimensional functions.

Definition 3.2. *Let I be a real interval, and $g : I \longrightarrow I$ a continuous map. We say that g is δ-strictly turbulent if there exist four constants $\beta_0 < \beta_1 < \gamma_0 < \gamma_1$ and $\delta > 0$ so that*

$$g(\beta_0) < \beta_0 - \delta < \gamma_1 + \delta < g(\beta_1),$$
$$g(\gamma_1) < \beta_0 - \delta < \gamma_1 + \delta < g(\gamma_0).$$

For simplicity in the presentation of our technique we consider the system

$$\begin{aligned} x_1(N) &= \alpha x_1(N-1) + (1-\alpha)(a_{11} f(x_1(N-k_{11})) + a_{12} f(x_2(N-k_{12}))) \\ x_2(N) &= \alpha x_2(N-1) + (1-\alpha)(a_{21} f(x_1(N-k_{21})) + a_{22} f(x_2(N-k_{22}))), \end{aligned} \tag{3.2}$$

and assume that (A1) and the following conditions hold:

(H1) $k = k_{12} = k_{21}$,
(H2) $1 \leqslant k_{ii} \leqslant k$ for $i = 1, 2$.

The general system (2.1) can be treated in a similar way, but the notations are more cumbersome. We point out that in this section we do not require assumptions (A2), (A3) on the matrix A.

Consider the map $F : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ associated with system (3.2):

$$F\begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ x_k \\ x_{k+1} \\ \vdots \\ x_{2k-1} \\ x_{2k} \end{pmatrix} = \begin{pmatrix} x_2 \\ \vdots \\ x_k \\ \alpha x_k + (1-\alpha)(a_{11} f(x_{k+1-k_{11}}) + a_{12} f(x_{k+1})) \\ x_{k+2} \\ \vdots \\ x_{2k} \\ \alpha x_{2k} + (1-\alpha)(a_{21} f(x_1) + a_{22} f(x_{2k+1-k_{22}})) \end{pmatrix}$$

For this map we have the following result.

Theorem 3.1. *Assume that (H1) and (H2) hold, and f is δ-strictly turbulent with parameters $\beta_0 < \beta_1 < \gamma_0 < \gamma_1$. There exists a constant $A > 0$ such that if*

$$\max\{\alpha, |a_{12} - 1|, |a_{21} - 1|, |a_{11}|, |a_{22}|\} < A, \tag{3.3}$$

then F^k has chaotic dynamics on two symbols relative to $N_0 = [\beta_0, \beta_1]^{2k}$ and $N_1 = [\gamma_0, \gamma_1]^{2k}$.

Proof. From the expression of F, it follows that for $\alpha = 0$, $a_{21} = a_{12} = 1$ and $a_{22} = a_{11} = 0$, we have

$$F^k(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{2k}) = (f(x_{k+1}), \ldots, f(x_{2k}), f(x_1), \ldots, f(x_k)).$$

It is clear that

$$G(x_1, \ldots x_k, x_{k+1}, \ldots, x_{2k}) = (x_{k+1}, \ldots, x_{2k}, x_1, \ldots x_k)$$

is a homeomorphism, $G(N_0) = N_0$, $G(N_1) = N_1$, and

$$(F^k \circ G)(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{2k}) = (f(x_1), \ldots, f(x_k), f(x_{k+1}), \ldots, f(x_{2k})).$$

Now using the continuity of F with respect to α and a_{ij}, we can obtain a positive constant A such that if (3.3) is satisfied then, for all $i = 1, \ldots, 2k$ and

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{2k}) \in [\beta_0, \beta_1]^{2k-1},$$

the following inequalities hold:

$$\left|(F^k \circ G)_i(x_1, \ldots, x_{i-1}, \beta_0, x_{1+i}, \ldots, x_{2k}) - f(\beta_0)\right| < \delta, \tag{3.4}$$

$$\left|(F^k \circ G)_i(x_1, \ldots, x_{i-1}, \beta_1, x_{1+i}, \ldots, x_{k2}) - f(\beta_1)\right| < \delta. \tag{3.5}$$

Analogously, for all $i = 1, \ldots, 2k$ and $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{2k}) \in [\gamma_0, \gamma_1]^{2k-1}$,

$$\left|(F^k \circ G)_i(x_1, \ldots, x_{i-1}, \gamma_0, x_{1+i}, \ldots, x_{2k}) - f(\gamma_0)\right| < \delta, \tag{3.6}$$

$$\left|(F^k \circ G)_i(x_1, \ldots, x_{i-1}, \gamma_1, x_{1+i}, \ldots, x_{2k}) - f(\gamma_1)\right| < \delta. \tag{3.7}$$

Once we have this set of estimations, we reason exactly as in the proof of theorem 4.2 in [26] to ensure the presence of chaotic dynamics for F^k relative to N_0 and N_1. The only difference is that we have to consider the linear maps

$$A(x) = (2x_{k+1}, \ldots, 2x_{2k}, 2x_1, \ldots, 2x_k)$$

and

$$A(x) = (-2x_{k+1}, \ldots, -2x_{2k}, -2x_1, \ldots, -2x_k)$$

instead of $A(x) = 2x$ and $A(x) = -2x$, respectively. □

Remark 3.1. If some iteration f^m is δ-strictly turbulent, then the conclusion of theorem 3.1 holds replacing F^k by F^{km}.

To guarantee the existence of chaotic dynamics in (3.2) for particular choices of parameters when f is δ-strictly turbulent we have to check conditions (3.4)–(3.7). We illustrate this fact with an example.

Example 3.1. Consider the system

$$\begin{aligned} x_1(N+1) &= \alpha x_1(N) + (1-\alpha)(a_{11} f(x_1(N)) + a_{12} f(x_2(N))) \\ x_2(N+1) &= \alpha x_2(N) + (1-\alpha)(a_{21} f(x_1(N)) + a_{22} f(x_2(N))), \end{aligned} \tag{3.8}$$

where

$$f(x) = 10\frac{1-\mathrm{e}^{-3x}}{1+\mathrm{e}^{-3x}} \cdot \frac{1-0.3\,\mathrm{e}^{5(|x|-1)}}{1+\mathrm{e}^{5(|x|-1)}}$$

is a non-monotonic function of the type of (2.8). We can check that f is 2.3-strictly turbulent with parameters $-0.2 < 0.5 < 0.7 < 1.7$. Next, it is easy to verify that conditions (3.4)–(3.7) hold for f provided the following inequalities are satisfied:

$$1.7\alpha + 6\left((1-\alpha)|a_{11}| + \alpha|a_{12}| + |a_{12} - 1|\right) < 2.3, \tag{3.9}$$

$$1.7\alpha + 6\left((1-\alpha)|a_{22}| + \alpha|a_{21}| + |a_{21} - 1|\right) < 2.3. \tag{3.10}$$

For example, to verify (3.4) for $i = 1$, take $x_2 \in [-0.2, 0.5]$. Then,

$$\begin{aligned} |F_1(x_2, -0.2) - f(-0.2)| &= |\alpha x_2 + (1-\alpha)(a_{11} f(x_2) + a_{12} f(-0.2)) - f(-0.2)| \\ &\leqslant 0.5\alpha + (1-\alpha)|a_{11} f(x_2)| + \alpha|a_{12} f(-0.2)| + |f(-0.2)(1-a_{12})|. \end{aligned}$$

Finally we apply (3.9) using that $|f(x)| \leqslant 6$, for all $x \in [-0.2, 0.5]$. The rest of the conditions are checked in the same way. We note that from conditions (3.9)–(3.10) it is clear that our method is robust under small perturbations of the parameters, since it works for the set of values of $\alpha, a_{11}, a_{12}, a_{21}, a_{22}$ for which those two inequalities hold.

Now we discuss some implications of our results for system (3.8). Specifically, we find two disjoint regions, namely

$$N_0 = [-0.2, 0.5]^2, \qquad N_1 = [0.7, 1.7]^2,$$

with the 'coin-tossing' property for F. For instance, if we take the bi-infinite sequence

$$(\ldots, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots)$$

(assume that the first 1 corresponds to the index 0 in the sequence), we can choose an initial point $x(0) = (x_1(0), x_2(0))$ so that the solution $\{x(N)\}_{N\in\mathbb{N}}$ of (3.8) starting at $x(0)$ satisfies

$$x(0) \in N_1,\ x(1) \in N_0,\ x(2) \in N_1,\ x(3) \in N_0,\ x(4) \in N_0,\ x(5) \in N_1, \ldots$$

As a direct consequence of proposition 3.1, we can also estimate the sensitive dependence. Given $\varepsilon > 0$, clearly

$$S_\varepsilon \leqslant \left\lceil \frac{(0.5+0.2)^2}{\varepsilon^2} \right\rceil := B.$$

Hence, there are two points $x_0, y_0 \in [-0.2, 0.5]^2$ such that $\|x_0 - y_0\| < \varepsilon$ and, for some number $j \in \{1, \ldots, N^*\}$, we have that $F^j(x_0) \in [-0.2, 0.5]^2$ and $F^j(y_0) \in [0.7, 1.7]^2$. Therefore,

$$\|F^j(x_0) - F^j(y_0)\| \geqslant 0.2.$$

Note that $\lceil \frac{\ln B}{\ln 2} \rceil + 1$ is an upper bound for N^*.

4. Discussion

In this paper we have given a detailed mathematical analysis of the dynamics of system (1.1), focused on the issues of global stability, multistability and the presence of chaotic dynamics. In this section we explain the mathematical results and discuss their practical relevance.

The most important feature of the analysis of global stability in section 2 is the fact that all results depend on the dynamics of the one-dimensional map (2.2), which is constructed from the activation function and the sum of the synaptic weights. In contrast with the literature, the problem of delay-independent stability in neural networks has been studied using the assumption of the monotone activation map, see [6, 12, 20, 28, 33, 37, 41, 42, 45, 50]. We point out that our approach enables us to obtain optimal conditions for non-monotonic activation functions. From a practical point of view, theorems 2.1–2.4 provide some noteworthy consequences: under excitatory connections, delays cannot destabilize the null solution for any activation function. Moreover, the same property arises for nontrivial equilibria in (1.1) when we restrict our attention to the regions $(0, +\infty)^n$ and $(-\infty, 0)^n$. An analogous phenomenon has been reported by several authors (see, for instance, [3] and [44, p 89]) in regard to continuous models of neural networks with sigmoid activation functions.

A crucial problem in artificial neural networks consists of studying synchronized/desynchronized behaviour in networks with identical neurons, see [5, 28, 44]. The scenario of identical neurons is translated mathematically assuming that system (1.1) is invariant under any permutation of indices. This problem leads to a discussion on when the set

$$\Delta = \{(x_1, \ldots, x_n) : x_i = x_j\}$$

is a global attractor for system (1.1). In the light of our results, under conditions (A1)–(A3) and (B1)–(B4), we can ensure that time delays do not produce asynchronous behaviours in system (2.1) in the regions $(0, +\infty)^n$ and $(-\infty, 0)^n$. Asynchronous patterns may be caused only by the activation function and the synaptic weights.

In section 3, we deal with complex dynamics in system (1.1). In our theorem 3.1, we provide easily verifiable sufficient conditions for the existence of chaotic solutions; for

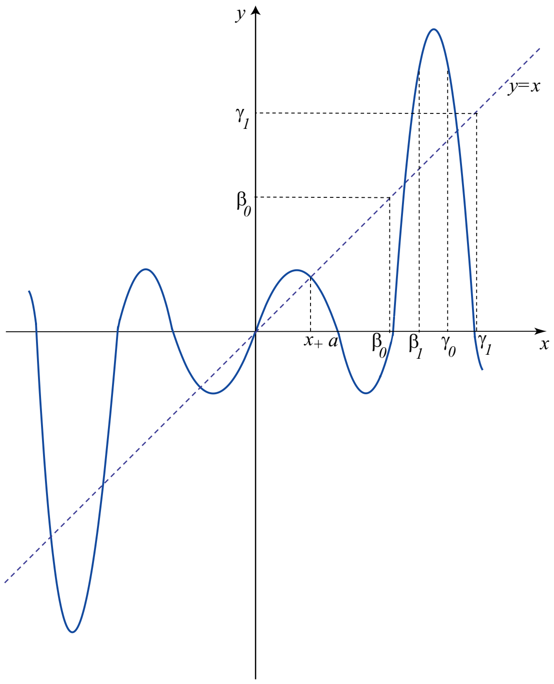

Figure 2. Graph of a map f for which system (2.1) may have stable equilibria with large basins of attraction, together with regions with complex dynamics. On the one hand, the equilibrium x_+ attracts all solutions of (2.1) starting in $(0, a)^n$; on the other hand, f is δ-strictly turbulent with parameters $\beta_0 < \beta_1 < \gamma_0 < \gamma_1$, which induces chaotic dynamics in (2.1) for suitable values of α and a_{ij}.

example, one of the consequences of this result is that an activation function with strong variations produces chaos in the network provided the internal decay of the neurons is small. This conclusion is of a different nature to some recent results in this direction; see, e.g., [4,36], where the authors consider excitatory–inhibitory networks and find conditions for when this model exhibits irregular and chaotic-like solutions.

Comparing with other previous work on complex behaviour in neural network models (see, e.g., [6, 14, 19, 22, 23, 42]), our results have the advantage of allowing us to estimate explicit parameter ranges where system (1.1) has chaotic dynamics, regions of initial data where the chaotic behaviour occurs, and sensitive dependence on the initial conditions (see example 3.1).

To conclude this section, we place emphasis on the possibility of the co-existence of chaos and stable equilibria in system (1.1). To illustrate this possibility, we combine theorems 2.4 and 3.1 for an activation function with the graph sketched in figure 2. The possibility of this co-existence phenomenon has been revealed in [19]; we stress that with our approach we can explicitly determine some regions with complex behaviour and others with simple dynamics.

Acknowledgments

EL was supported in part by the Spanish Government and FEDER, grant MTM2010–14837. A R-H was supported by the Spanish Government, grant MTM2011–23652 and is currently supported by the ERC Starting Investigator grant no. 259559. The second author thanks the Departamento de Matemática Aplicada II of the University of Vigo (Spain) for its kind hospitality.

References

[1] Aulbach B and Kieninger B 2001 On three definitions of chaos *Nonlinear Dyn. Syst. Theory* **1** 23–37
[2] Baldi P and Atiya A F 1994 How delays affect neural dynamics and learning *IEEE Trans. Neural Netw.* **5** 612–21
[3] Bélair J, Campbell S A and van den Driessche P 1996 Frustration, stability and delay-induced oscillations in a neural network model *SIAM J. Appl. Math.* **56** 245–55
[4] Best J, Park C, Terman D and Wilson C 2007 Transitions between irregular and rhythmic firing patterns in excitatory-inhibitory neuronal networks *J. Comput. Neurosci.* **23** 217–35
[5] Bungay S D and Campbell S A 2007 Patterns of oscillation in a ring of identical cells with delayed coupling *Int. J. Bifur. Chaos Appl. Sci. Eng.* **17** 3109–25
[6] Cheng C 2010 Coexistence of multistability and chaos in a ring of discrete neural networks with delays *Int. J. Bifur. Chaos Appl. Sci. Eng.* **20** 1119–36
[7] Chua L O 1998 *CNN: A Paradigm for Complexity* (Singapore: World Scientific)
[8] Clark C W 1976 A delayed recruitment model of population dynamics with an application to baleen whale populations *J. Math. Biol.* **3** 381–91
[9] Coppel W A 1955 The solution of equations by iteration *Proc. Camb. Phil. Soc.* **51** 41–3
[10] El-Morshedy H A and Jiménez López V 2008 Global attractors for difference equations dominated by one-dimensional maps *J. Diff. Eqns Appl.* **14** 391–410
[11] El-Morshedy H A and Liz E 2006 Globally attracting fixed points in higher order discrete population models *J. Math. Biol.* **53** 365–84
[12] Folias S E and Ermentrout G B 2012 Bifurcations of stationary solutions in an interacting pair of E–I neural fields *SIAM J. Appl. Dyn. Syst* **11** 895–938
[13] Foss J, Longtin A, Mensour B and Milton J 1996 Multistability and delayed recurrent loops *Phys. Rev. Lett.* **76** 708–11
[14] Guo S 2013 Zero singularities in a ring network with two delays *Z. Angew. Math. Phys.* **64** 201–22
[15] Hahnloser R H, Sarpeshkar R, Mahowald M A, Douglas R J and Seung S 2000 Digital selection and analogue amplification coexist in a cortex-inspired silicon circuit *Nature* **405** 947–51
[16] Hopfield J 1982 Neural networks and physical systems with emergent collective computational abilities *Proc. Natl Acad. Sci.* **79** 2554–8
[17] Hopfield J 1984 Neurons with graded response have collective computational properties like those of two-state neurons *Proc. Natl Acad. Sci.* **81** 3088–92
[18] Horn R A and Johnson C R 1990 *Matrix Analysis* (Cambridge: Cambridge University Press)
[19] Huang Y and Zou X 2005 Co-existence of chaos and stable periodic orbits in a simple discrete neural network *J. Nonlinear Sci.* **15** 291–303
[20] Huang Z, Wang X and Feng C 2010 Multiperiodicity of periodically oscillated discrete-time neural networks with transient excitatory self-connections and sigmoidal nonlinearities *IEEE Trans. Neural Netw.* **21** 1643–55
[21] Ivanov A F and Sharkovsky A N 1992 Oscillations in singularly perturbed delay equations *Dyn. Rep. Expositions Dyn. Syst. (N.S.)* **1** 164–224
[22] Kaslik E and Balint S 2008 Chaotic dynamics of a delayed discrete-time Hopfield network of two nonidentical neurons with no self-connections *J. Nonlinear Sci.* **18** 415–32
[23] Kaslik E and Balint S 2009 Complex and chaotic dynamics in a discrete-time-delayed Hopfield neural network with ring architecture *Neural Netw.* **22** 1411–28
[24] Liz E 2009 Global stability and bifurcations in a delayed discrete population model *Int. J. Qual. Theory Diff. Eqns Appl.* **3** 66–80
[25] Liz E and Röst G 2009 On the global attractor of delay differential equations with unimodal feedback *Discrete Contin. Dyn. Syst.* **24** 1215–24
[26] Liz E and Ruiz-Herrera A 2012 Chaos in discrete structured population models *SIAM J. Appl. Dyn. Syst.* **11** 1200–14
[27] Liz E and Ruiz-Herrera A 2013 Attractivity, multistability, and bifurcation in delayed Hopfield's model with non-monotonic feedback *J. Diff. Eqns* **255** 4244–66
[28] Ly C and Ermentrout G B 2009 Synchronization dynamics of two coupled neural oscillators receiving shared and unshared noisy stimuli *J. Comput. Neurosci.* **26** 425–43

[29] Ma J and Wu J 2009 Multistability and gluing bifurcation to butterflies in coupled networks with non-monotonic feedback *Nonlinearity* **22** 1383–412

[30] Mallet-Paret J and Nussbaum R 1986 Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation *Ann. Mat. Pura Appl.* **145** 33–128

[31] Marcus C M and Westervelt R M 1989 Stability of analog neural networks with delay *Phys. Rev.* A **39** 347–59

[32] Medio A, Pireddu M and Zanolin F 2009 Chaotic dynamics for maps in one and two dimensions: a geometrical method and applications to economics *Internat. J. Bifur. Chaos Appl. Sci. Eng.* **19** 3283–309

[33] Mohamad S and Gopalsamy K 2000 Dynamics of a class of discrete-time neural networks and their continuous-time counterparts *Math. Comput. Simul.* **53** 1–39.

[34] Morita M 1993 Associative memory with non-monotone dynamics *Neural Netw.* **6** 115–23

[35] Morita M, Yoshizawa S and Nakano K 1990 Memory of correlated patterns by associative neural networks with improved dynamics *Proc. INNC'90 (Paris, France)* vol 2 pp 868–71

[36] Park C and Terman D 2010 Irregular behavior in an excitatory-inhibitory neuronal network *Chaos* **20** 023122

[37] Peng M and Yang X 2010 New stability criteria and bifurcation analysis for nonlinear discrete-time coupled loops with multiple delays *Chaos* **20** 013125

[38] Singer D 1978 Stable orbits and bifurcation of maps of the interval *SIAM J. Appl. Math.* **35** 260–7

[39] Uhlhaas P J and Singer W 2006 Neural synchrony in braid disorders: relevance for cognitive dysfunctions and pathophysiology *Neuron* **52** 155–68

[40] Tank D and Hopfield J 1986 Simple 'neural' optimization networks: an A/D converter, signal decision circuit and a linear programming circuit *IEEE Trans. Circuits Syst.* **33** 533–41

[41] van den Driessche P and Zou X 1998 Global attractivity in delayed Hopfield neural network models *SIAM J. Appl. Math.* **58** 1878–90

[42] Wang G and Peng M 2009 Rich oscillation patterns in a simple discrete-time delayed neuron network and its linear control *Int. J. Bifur. Chaos Appl. Sci. Eng.* **19** 2993–3004

[43] Wang L and Zou X 2004 Capacity of stable periodic solutions in discrete-time bidirectional associative memory neural networks *IEEE Trans. Circuits Syst. II: Express Briefs* **51** 315–9

[44] Wu J 2001 *Introduction to Neural Dynamics and Signal Transmission Delay* (*de Gruyter Series in Nonlinear Analysis and Applications* vol 6) (Berlin: de Gruyter)

[45] Wu J and Zhang Y 2004 A simple delayed neural network for associative memory with large capacity *Discrete Contin. Dyn. Syst. Ser.* B **4** 851–63

[46] Wu J, Zhang R Y and Zou X 2004 Multiple periodic patterns via discrete neural nets with delayed feedback loops *Int. J. Bifur. Chaos Appl. Sci. Eng.* **14** 2915–23

[47] Yi T and Zou X 2010 Maps dynamics versus dynamics of associated delay reaction–diffusion equation with a Neumann condition *Proc. R. Soc. Lond.* A **466** 2955–73

[48] Yi T and Zou X 2011 Global dynamics of a delay differential equation with spatial non-locality in an unbounded domain *J. Diff. Eqns* **251** 2598–611

[49] Zgliczyński P and Gidea M 2004 Covering relations for multidimensional dynamical systems *J. Diff. Eqns* **202** 32–58

[50] Zhou Z and Wu J 2002 Attractive periodic orbits in nonlinear discrete-time neural network with delayed feedback *J. Diff. Eqns Appl.* **8** 467–83