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Prediction-based control of chaos and a dynamic Parrondo's paradox

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ABSTRACT

We analyze the stabilization of an unstable periodic orbit (UPO) by periodic prediction-based control (PBC). We rigorously prove that, for 2-periodic orbits, a pulse strategy reduces the necessary control strength to stabilize the UPO. Moreover, we find that in some cases the periodic control prevents some undesirable effects induced by the PBC method. In this way, we provide an example of a dynamic Parrondo's paradox: the switching between two undesirable dynamics results in a nicely periodic dynamic behavior.

1. Introduction

Control of chaotic behavior has become a very important issue in many areas [1]. In this Letter, we further develop the understanding and analytic insight into prediction-based control (PBC) of chaos introduced by Ushio and Yamamoto [2]. In particular, we deal with the stabilization of an unstable periodic orbit of a chaotic one-dimensional discrete dynamical system

$$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots, \tag{1.1}$$

when PBC is applied periodically, in the form of pulses.

Periodic control schemes have been proposed for (1.1) using various control techniques, including proportional feedback control [3–6], and delayed feedback control [7–10]. We are particularly interested in biological control, where pulse strategies are especially important; for example, due to seasonal interventions in population dynamics, periodic migrations, or climatic periodic signals [11–13].

The analysis of pulse stabilization using PBC was initiated in [14]; the main conclusion is that, if K is a fixed point of f and $f'(K) < -1$, then, for an arbitrary period $m > 0$, there is an open real interval I_m such that, for any $\alpha \in I_m$, K is asymptotically stable for the pulse scheme

$$x_n = \begin{cases} f(x_{n-1}), & \text{if } n \notin m\mathbb{N}, \\ f(x_{n-1}) - \alpha(f(x_{n-1}) - x_{n-1}), & \text{if } n \in m\mathbb{N}. \end{cases} \tag{1.2}$$

However, the interval I_m gets smaller as m increases, and the length of I_m tends to zero as m tends to infinity.

In this Letter, we study the periodic stabilization of nontrivial periodic orbits of (1.1) based on the PBC method. Contrary to what happens with unstable fixed points, for stabilizing a 2-cycle the pulse scheme requires less control strength than the classic PBC method.

Moreover, we show that the pulse scheme may avoid some undesirable effects induced by the control; indeed, in some situations, the control action stabilizes a periodic orbit but, at the same time, it gives rise to a new positive equilibrium in such a way that initial conditions below this point are driven to zero by the controlled system. In population dynamics, this type of behavior is referred to as an Allee effect [15,16]; Allee effects increase the risk of extinction and thus they are crucial in conservation and management. The possibility of PBC to lead to Allee effects has already been reported in [17].

We prove that a 2-periodic control keeps the stabilization properties and prevents the Allee effect. Since the pulse scheme can be seen as a periodic switching between the map f of the original system and the controlled map, the beneficial effects of applying the periodic feedback may be interpreted as a dynamic Parrondian game [18]; that is to say, an alternation of different dynamics with undesirable properties can give rise to a desirable dynamic.

As far as we know, the first extension of the famous Parrondo's paradox to dynamical systems is due to Almeida et al. [19] (see also [20]). There, the authors show that a combination of two chaotic systems may produce order; thus, they find a "chaos + chaos = order" phenomenon analogous to the Parrondo's paradox "losing + losing = winning". Our example goes in the line

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of the recent note [21], where it is argued that in some simple population models a switching between two bad environmental conditions may drive the population to a desirable behavior.

2. Results and discussion

2.1. Main results

We consider the prediction-based feedback control introduced in [2]. The main idea of this method to stabilize an unstable T-periodic orbit

$$P = \{p_1, p_2, \dots, p_T\}$$

of (1.1) consists of determining the control input u_n by the difference between the predicted states and the current states, that is,

$$u_n = -\alpha\bigl(f^T(x_n) - x_n\bigr),$$

where α is a real parameter. Thus, the PBC scheme writes

$$x_n = f(x_{n-1}) - \alpha\bigl(f^T(x_{n-1}) - x_{n-1}\bigr). \tag{2.1}$$

The absolute value of α represents the control strength, and the term $f^T(x_n)$ is used as a prediction of x_{n+T}. As usual, f^T denotes the corresponding power of f under composition, that is,

$$f^T = \underbrace{f \circ \cdots \circ f}_{T}.$$

It is clear that P is also a T-periodic orbit of (2.1). By assumption, the multiplier $d = f'(p_1)f'(p_2)\cdots f'(p_T)$ satisfies the inequality $|d| \geqslant 1$, and a sufficient condition for the local asymptotic stability of P in the controlled system for a given value of α is

$$\bigl|g'_\alpha(p_1)g'_\alpha(p_2)\cdots g'_\alpha(p_T)\bigr| < 1, \tag{2.2}$$

where $g_\alpha(x) := f(x) - \alpha(f^T(x) - x)$.

We propose the periodic prediction-based control scheme

$$x_n = \begin{cases} f(x_{n-1}), & \text{if } n \notin T\mathbb{N}, \\ f(x_{n-1}) - \alpha(f^T(x_{n-1}) - x_{n-1}), & \text{if } n \in T\mathbb{N}. \end{cases} \tag{2.3}$$

Again, we immediately check that P is a T-periodic solution of the periodic difference equation (2.3). Moreover, its stability properties depend on the period map $h_\alpha = g_\alpha \circ f^{T-1}$, as the following result shows.

Proposition 2.1. *Let $P = \{p_1, p_2, \dots, p_T\}$ be a T-periodic orbit of (1.1). If $|h'_\alpha(p_i)| < 1$ for some $i \in \{1, 2, \dots, T\}$, then P is locally asymptotically stable for the pulse scheme (2.3).*

We prove Proposition 2.1 in Appendix A.

For the case $T = 2$, we find the set of real values α for which (2.3) is asymptotically stable, and we prove, using Proposition 2.1, that the periodic scheme works better than the usual PBC method in the following sense: if (2.1) stabilizes $\{p_1, p_2\}$ for some $\alpha \in \mathbb{R}$, then (2.3) also stabilizes $\{p_1, p_2\}$ for the same value of α. We need to impose an additional condition, but we demonstrate that this is not a restriction for many usual maps f (see Propositions A.1, A.2 and A.3 in Appendix A).

In the next subsection, we present an example that illustrates these results and points out other advantages of using a periodic control scheme. In the light of the example, we discuss a dynamic Parrondo's paradox associated to prediction-based control.

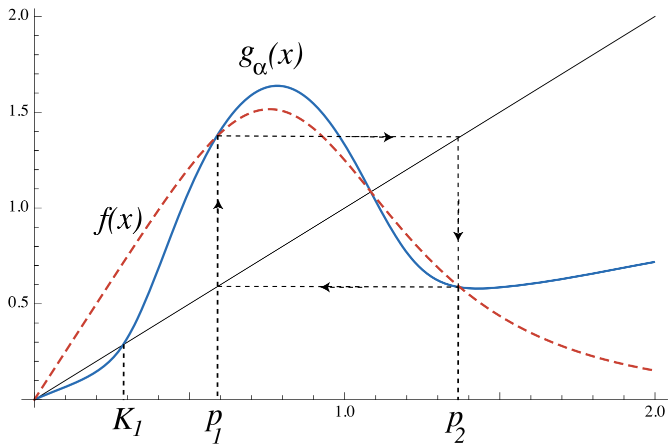

Fig. 1. Graphs of $f(x) = 2.5x/(1+x^5)$ (dashed line) and $g_\alpha(x) = f(x) - \alpha(f^2(x) - x)$ (solid line) for $\alpha = 0.35$. The 2-cycle $\{p_1, p_2\}$ is stabilized by the control (2.5), but a new fixed point K_1 of g_α appears, giving place to an Allee effect: initial conditions below K_1 are driven to zero by g_α.

2.2. Case study

Consider a pulse scheme to stabilize a 2-periodic orbit of the generalized Beverton–Holt map [22]

$$f(x) = \frac{2.5x}{1 + x^5}. \tag{2.4}$$

The map f is chaotic, and has an infinite number of unstable periodic orbits. We aim to stabilize the 2-periodic orbit $\{p_1, p_2\} = \{0.5864, 1.3714\}$, with $f'(p_1)f'(p_2) = -2.1245$. We begin applying the PBC scheme (2.1), that can be written in the form

$$x_n = g_\alpha(x_{n-1}), \tag{2.5}$$

where $g_\alpha(x) = f(x) - \alpha(f^2(x) - x)$. We can check that $\{p_1, p_2\}$ is stabilized by (2.5) if

$$\alpha \in D_1 = (-0.6044, -0.3789) \cup (0.3039, 0.5294). \tag{2.6}$$

However, for $\alpha > 0$, a drawback of this method is that the control (2.5) makes the trivial equilibrium asymptotically stable. Indeed,

$$g'_\alpha(0) = f'(0) - \alpha\bigl(f'(0)^2 - 1\bigr) < 1 \iff \alpha > \frac{1}{1 + f'(0)}, \tag{2.7}$$

and therefore (2.5) displays a strong Allee effect if $\alpha > 2/7 = 0.285714$. For these values of α, the map g_α has two positive fixed points K_1, K_2, where K_2 is a fixed point of f but K_1 is not. The new equilibrium K_1 represents a threshold population density below which the population growth is negative, resulting in extinction [16].

For $\alpha \in (0.3039, 0.5294)$, system (2.5) exhibits bistability: the stable 2-periodic point $\{p_1, p_2\}$ coexists with the stable fixed point $x = 0$. Hence, a good effect of the control intervention is the suppression of chaos and stabilization of a 2-periodic orbit. However, the pitfall is that, while a population model governed by f is permanent (the chaotic attractor is bounded away from zero), this property is lost when we apply the PBC method: an Allee effect causes extinction for low population densities.

In Fig. 1, we show the graph of g_α for $\alpha = 0.35$, which has the 2-periodic attractor $\{p_1, p_2\}$ and the fixed points $K_0 = 0$, $K_1 = 0.2854$, $K_2 = 1.0844$. The interval $(0, K_1)$ is the basin of attraction of K_0. This situation of bistability is observable in the bifurcation diagram plotted in Fig. 2(a).

Now we apply the periodic scheme (2.3) with $T = 2$, that is,

$$x_n = \begin{cases} f(x_{n-1}), & \text{if } n \notin 2\mathbb{N}, \\ g_\alpha(x_{n-1}), & \text{if } n \in 2\mathbb{N}. \end{cases} \tag{2.8}$$

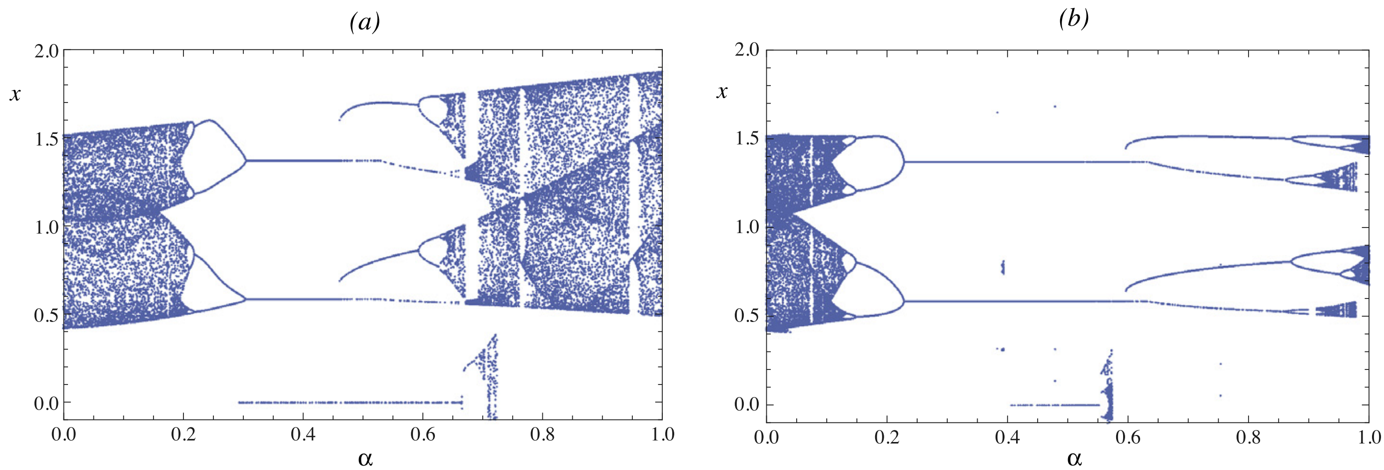

Fig. 2. Bifurcation diagrams of the controlled systems with and without pulses for $\alpha \in [0,1]$. (a) corresponds to the control scheme (2.5): even when it is stable, the 2-periodic orbit coexists with the attracting trivial equilibrium; (b) corresponds to the periodic control (2.8): the 2-periodic orbit of f seems to be globally attracting for $\alpha \in (0.2278, 0.3828)$. The systems are iterated 400 times for a random initial condition $x_0 \in [0,2]$, and only the last 20 iterations are plotted. The values of α are picked up on the interval $[0,1]$, with a step $\varepsilon = 0.0005$.

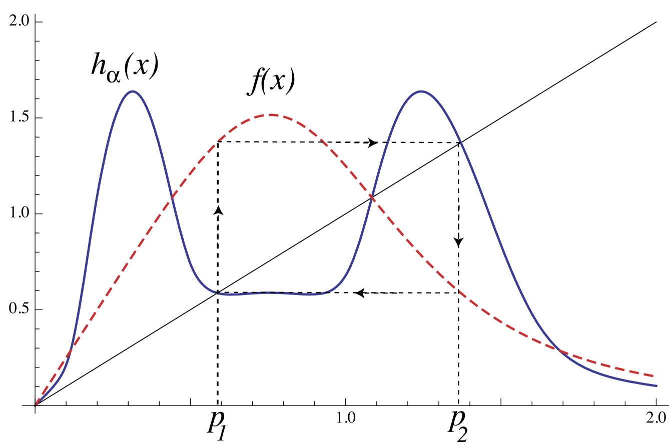

Fig. 3. Graphs of $f(x) = 2.5x/(1+x^5)$ (dashed line) and the period map $h_\alpha(x) = g_\alpha(f(x))$ (solid line) for $\alpha = 0.35$. p_1 is a stable fixed point of h_α and therefore the periodic scheme (2.8) stabilizes the 2-cycle $\{p_1, p_2\}$ of f. In this case, the system is permanent under the control scheme (hence, the Allee effect is prevented). Moreover, our numerical experiments suggest that $\{p_1, p_2\}$ attracts all positive initial conditions except the equilibrium.

Using Proposition A.1, we know that (2.8) stabilizes $\{p_1, p_2\}$ for

$\alpha \in D = (-0.7434, -0.2675) \cup (0.2278, 0.6331)$.

We check that D contains the set D_1 given in (2.6), in agreement with Proposition A.2. On the other hand, the equilibrium $x = 0$ is asymptotically stable for the period map $h_\alpha = g_\alpha \circ f$ if $h'_\alpha(0) < 1$, which is equivalent to $\alpha > 1/f'(0)$. Comparing this condition with (2.7), it is clear that the Allee effect is induced in the periodic scheme for larger values of α than in the control applied at every step.

For our case study, in contrast with (2.5), system (2.8) stabilizes the 2-periodic orbit $\{p_1, p_2\}$ and is permanent for $\alpha \in (0.2278, 0.4)$. Moreover, the 2-periodic cycle $\{p_1, p_2\}$ seems to attract every positive initial condition different from the equilibrium K_2 for $\alpha \in (0.2278, \alpha^*)$, where $\alpha^* \approx 0.3828$; see the bifurcation diagram in Fig. 2(b). Fig. 3 shows the graph of the period map h_α for $\alpha = 0.35$. The strong Allee effect observed in the PBC scheme (2.5) is prevented; instead, there is a robust 2-periodic global attractor.

This example shows how a pulse strategy can be beneficial: on the one hand, the control strength necessary to stabilize the 2-periodic orbit using the periodic scheme (2.8) is smaller than the corresponding one for the usual PBC control (2.5). On the other hand, the periodic control helps to prevent undesirable effects of the usual PBC control. This phenomenon can be interpreted as a dynamic Parrondian game, that is, the combination of two dynamics which yield undesired behaviors can give rise to a desirable behavior. Namely, in our example, we find the following situation for a range of values of the control parameter α:

- the dynamics of f is not good because it has a chaotic attractor. This fact means that long-term predictions are not possible;
- the dynamics of g_α are also not good because they exhibit an Allee effect. In population models, this effect induces a risk of extinction;
- the compositions $h_\alpha = g_\alpha \circ f$ resulting from switching between f and g_α exhibit good dynamics, because the system has a 2-periodic global attractor.

As argued in [23], paradoxical outcomes of the combination of two component systems rely on hidden correlations between them; in our example, the switch between f and g_α keeps the stability of the 2-periodic orbit $\{p_1, p_2\}$ and prevents the stabilization of the trivial equilibrium, thus avoiding the Allee effect.

In summary, we found a new and interesting example of Parrondo's paradox in nonlinear dynamics; for related results, see [19–21,23] and references therein.

3. Conclusions

Quoting Williams and Hastings [23], "*counterintuitive dynamics of various biological phenomena occur when composite system dynamics differ qualitatively from that of their component systems*". If the composition of two undesirable dynamics results in a desirable outcome, then the resulting dynamic is referred to as a dynamic Parrondo's paradox [18–20]. In the recent note [21], Peacock-López pointed out that a periodic pulse technique for controlling chaos may be described as a Parrondian game. In this direction, we have found an illustrative example while exploring a periodic prediction-based control. This finding underlines the importance of improving the analytic insight into different methods of chaos control, one of the issues emphasized by Schöll and Schuster in the preface of the Handbook of Chaos Control [1].

The findings of this Letter are especially relevant in control of chaotic populations, since one of the aims of control in populations is preventing extinction [11,24]. We showed that PBC may suppress chaos by stabilizing a 2-periodic cycle, but, at the same time, it may induce an Allee effect, and therefore a high risk of extinction if population density falls below a threshold level. The important and somehow paradoxical message of our study is that the use of a periodic control, instead of intervention at every step, may solve this drawback. Actually, a combination of two dynamics with a high risk of extinction can lead to a robust 2-periodic attractor. Notice that, since the convergence seems to be global, the periodically controlled system should remain permanent even in the presence of noise.

An additional and surprising conclusion is that the necessary control strength to stabilize a 2-periodic orbit is smaller with a pulse strategy. Thus, the control effort is reduced in two directions: decreasing both the strength and the frequency of interventions. We defined the control strength as the absolute value of the control parameter α; a different way of measuring the control effort consists of estimating the cost in terms of interventions, that is, the amount of the state variable added or removed over a number of generations. This aspect has been investigated in [25] for various control methods aiming to stabilize the positive equilibrium of a Ricker map. Extending this study to nontrivial periodic orbits and pulse strategies is an interesting future direction.

Finally, we stress the fact that our findings are supported by rigorous mathematical results (proved in Appendix A), and that they apply to a wide class of relevant families of one-dimensional maps usually employed in mathematical modeling.

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Appendix A

In this appendix we state and prove the stability results discussed in Section 2. We begin with the proof of Proposition 2.1.

Proof of Proposition 2.1. Recall that $g_\alpha(x) = f(x) - \alpha(f^T(x) - x)$ and $h_\alpha(x) = g_\alpha(f^{T-1}(x))$.

Any solution $\{x_n\}$ of the pulse scheme (2.3) satisfies that $x_{nT} = h_\alpha^n(x_0)$ and $x_{nT+i} = f^i(h_\alpha^n(x_0))$, for all $n \geqslant 1$ and $i = 1, 2, \dots, T-1$. Moreover, $h_\alpha(p_i) = p_i$ for all $i = 1, 2, \dots, T$.

Without restricting the generality, we suppose that $|h'_\alpha(p_1)| < 1$. Then p_1 is a local attractor fixed point for the system $y_{n+1} = h_\alpha(y_n)$, and every solution $\{x_n\}$ of (2.3) with x_0 belonging to the attraction basin of p_1 satisfies

$$\lim_{n\to\infty} x_{nT} = p_1,$$

$$\lim_{n\to\infty} x_{nT+i} = \lim_{n\to\infty} f^i(x_{nT}) = f^i\Big(\lim_{n\to\infty} x_{nT}\Big) = f^i(p_1) = p_{i+1},$$

for all $i = 1, 2, \dots, T-1$. Hence the T-periodic orbit $\{p_1, p_2, \dots, p_T\}$ is attracting for the pulse scheme (2.3). By a well-known result (see, e.g., [26, Theorem 4.7, p. 182]), the T-periodic orbit is stable, and therefore locally asymptotically stable. $\square$

In the following, we consider the case $T = 2$. Thus, we assume that $\{p_1, p_2\}$ is an unstable 2-periodic orbit of f. As in the general case, given $\alpha \in \mathbb{R}$, we denote $g_\alpha(x) = f(x) - \alpha(f^2(x) - x)$ and $h_\alpha(x) = g_\alpha \circ f(x)$.

Hence, the periodic PBC scheme is

$$x_n = \begin{cases} f(x_{n-1}), & \text{if } n \notin 2\mathbb{N}, \\ g_\alpha(x_{n-1}), & \text{if } n \in 2\mathbb{N}. \end{cases} \tag{A.1}$$

First we use Proposition 2.1 to give the explicit values of α for which (A.1) stabilizes $\{p_1, p_2\}$.

Proposition A.1. *Assume that*

$$f'(p_1) > 0 \quad \text{and} \quad f'(p_2) < 0. \tag{A.2}$$

A parameter $\alpha \in \mathbb{R}$ satisfies $|h'_\alpha(p_1)| < 1$ or $|h'_\alpha(p_2)| < 1$ if and only if

$$\alpha \in D = \left(\frac{1}{f'(p_2)}, \frac{d+1}{f'(p_2)(d-1)}\right) \cup \left(\frac{d+1}{f'(p_1)(d-1)}, \frac{1}{f'(p_1)}\right),$$

where $d = f'(p_1)f'(p_2) < -1$.

Proof. Notice that

$$\begin{aligned} h'_\alpha(p_1) &= g'_\alpha(p_2)f'(p_1) = \big(f'(p_2) - \alpha(d-1)\big)f'(p_1) \\ &= d - \alpha f'(p_1)(d-1), \end{aligned} \tag{A.3}$$

$$\begin{aligned} h'_\alpha(p_2) &= g'_\alpha(p_1)f'(p_2) = \big(f'(p_1) - \alpha(d-1)\big)f'(p_2) \\ &= d - \alpha f'(p_2)(d-1). \end{aligned} \tag{A.4}$$

Since $f'(p_1) > 0$, we deduce from equality (A.3) that $|h'_\alpha(p_1)| < 1$ is equivalent to say that $\alpha \in (\frac{d+1}{f'(p_1)(d-1)}, \frac{1}{f'(p_1)})$. On the other hand, using that $f'(p_2) < 0$, we deduce from (A.4) that $|h'_\alpha(p_2)| < 1$ is equivalent to say that $\alpha \in (\frac{1}{f'(p_2)}, \frac{d+1}{f'(p_2)(d-1)})$. $\square$

Our next result shows that if the classical PBC method stabilizes $\{p_1, p_2\}$ for some $\alpha \in \mathbb{R}$, and (A.2) holds, then (A.1) also stabilizes $\{p_1, p_2\}$ for the same value of α.

Proposition A.2. *If (A.2) holds and $\alpha \in \mathbb{R}$ satisfies $|g'_\alpha(p_1)g'_\alpha(p_2)| \leqslant 1$, then $\alpha \in D$.*

Proof. First, we study the case $\alpha > 0$. Since $f'(p_1) > 0$ and $d \leqslant -1$, we obtain

$$\begin{aligned} \big|h'_\alpha(p_1)\big| &= \big|g'_\alpha(p_2)f'(p_1)\big| < \big|g'_\alpha(p_2)\big(f'(p_1) - \alpha(d-1)\big)\big| \\ &= \big|g'_\alpha(p_2)g'_\alpha(p_1)\big| < 1, \end{aligned}$$

and we deduce that $\alpha \in D$ by Proposition A.1.

It remains to consider the case $\alpha < 0$. Using that $f'(p_2) < 0$ and $d \leqslant -1$, we get

$$\begin{aligned} \big|h'_\alpha(p_2)\big| &= \big|g'_\alpha(p_1)f'(p_2)\big| < \big|g'_\alpha(p_1)\big(f'(p_2) - \alpha(d-1)\big)\big| \\ &= \big|g'_\alpha(p_1)g'_\alpha(p_2)\big| < 1, \end{aligned}$$

and we obtain again from Proposition A.1 that $\alpha \in D$. $\square$

We conclude this appendix showing that for the family of S-unimodal maps, that is, unimodal maps with negative Schwarzian derivative (cf. [27, Definition 2]), condition (A.2) is not restrictive. We recall that this family of maps contains the generalized Beverton–Holt map used in Section 2 and other usual functions such as the quadratic family $Q_\lambda(x) = \lambda x(1-x)$ and the Ricker family $R_{\lambda,\beta}(x) = \lambda x e^{-\beta x}$. As already noticed by Singer [28], most of the commonly studied one-dimensional maps are S-unimodal.

Proposition A.3. *Assume that f is S-unimodal and $\{p_1, p_2\}$ is an unstable 2-periodic orbit of f. Then condition (A.2) holds.*

Proof. We can assume that $f'(p_1)f'(p_2) \neq 0$ because otherwise $\{p_1, p_2\}$ is stable. Next, it is obvious that $f'(p_1) > 0$, $f'(p_2) > 0$ is impossible for a 2-periodic orbit. Finally, suppose that $f'(p_1) < 0$,

$f'(p_2) < 0$. Then f restricted to the invariant interval $[p_1, p_2]$ is a decreasing function, and it follows from [29, Theorem 1] that $\{p_1, p_2\}$ is attracting; thus we arrive at a contradiction that completes the proof. □

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