Crossed products of crossed modules of Hopf monoids in a braided setting

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Based in a joint work with J.N. Alonso Álvarez and J.M. Fernández Vilaboa

Rings, modules, and Hopf algebras

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Outline

The setting

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  Without loss of generality, by the coherence theorems, we can assume the monoidal structure of $\mathcal{C}$ strict. Then, in this talk, we omit explicitly the associativity and unit constraints.

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- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

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- If $f, g : C \to A$ are morphisms, $f * g$ denotes the convolution product.

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_C.$$

- If $\mathcal{C}$ is braided with braiding c, a bimonoid H is a monoid (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that η_H and μ_H are morphisms of comonoids (equivalently, ε_H and δ_H are morphisms of monoids).

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- If moreover there exists a morphism

$$\lambda_H : H \to H$$

(called the antipode of H) such that

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- If H and G are Hopf monoids, $f : H \to G$ is a morphism of Hopf monoids if it is a monoid and comonoid morphism. In this case

$$\lambda_G \circ f = f \circ \lambda_H.$$

- Let H be a Hopf monoid. An object M in $\mathcal{C}$ is said to be a left H-module if there is a morphism $\phi_M : H \otimes M \to M$ in $\mathcal{C}$ satisfying that

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$$\phi_B \circ (H \otimes \eta_B) = \varepsilon_H \otimes \eta_B, \quad \phi_B \circ (H \otimes \mu_B) = \mu_B \circ (\phi_B \otimes \phi_B) \circ (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes B \otimes B),$$

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- If B is a comonoid and ε_B and δ_B are left H-module morphisms, i.e.,

$$\varepsilon_B \circ \phi_B = \varepsilon_H \otimes \varepsilon_B, \quad \delta_B \circ \phi_B = (\phi_B \otimes \phi_B) \circ \delta_{H \otimes B},$$

  where $\delta_{H \otimes B} = (H \otimes c_{H,B} \otimes B) \circ (\delta_H \otimes \delta_B)$, (B, ϕ_B) is said to be a left H-module comonoid.

- If H is a Hopf monoid, B a monoid and $f : H \rightarrow B$ a monoid morphism, the adjoint action of H on B associated to f is defined as

$$ad_{f,B} = \mu_B \circ (\mu_B \otimes B) \circ (f \otimes B \otimes (f \circ \lambda_H)) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B).$$

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  Then (B, ϕ_B) is a left H-module monoid with $\phi_B = ad_{f,B}$.
- In particular, if $B = H$ and $f = id_H$ the action defined above (called the adjoint action of H) is the following:

$$ad_{id_H,H} = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

  In what follows we will denote this action by ad_H.

Some definitions of crossed modules of Hopf monoids

First definition

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Let B, H be groups and let $\beta : B \to H$ be a group morphism. Let

$$\phi_B : H \times B \to B, \quad \phi_B(h, b) = {}^{h}b$$

be an action of H over B. The triple

$$\mathbf{B_H} = (B, H, \beta)$$

is a crossed module of groups if the following identities hold:

(i) $\beta({}^{h}b) = h\beta(b)h^{-\mathbf{1}}$.

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In this setting, $\mathbf{H_H} = (H, H, id_H)$ is an example of is a crossed module of groups with $\phi_H(h, b) = hbh^{-1}$ (the adjoint action).

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In this setting, $\mathbf{H_H} = (H, H, id_H)$ is an example of is a crossed module of Hopf monoids for $\phi_H = ad_H$ because $\mathcal{C}$ is symmetric and H is cocommutative.

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Assume that $Vect_{\mathbb{K}}$ is a category of vector spaces over a field $\mathbb{K}$. Let H, B be Hopf monoids (algebras) in $Vect_{\mathbb{K}}$.

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Assume that $Vect_{\mathbb{K}}$ is a category of vector spaces over a field $\mathbb{K}$. Let H, B, be Hopf monoids (algebras) in $Vect_{\mathbb{K}}$.

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In this setting, if the antipode of H is an isomorphism, $\mathbf{H_H} = (H, H, id_H)$ is an example of is a crossed module for $\phi_H = ad_H$ because (iii) holds and $Vect_{\mathbb{K}}$ is symmetric.

A new definition

In this point the category $\mathcal{C}$ is braided with braiding c

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Proposition

Let H and B be Hopf monoids, and let $f : H \to B$ be a bimonoid morphism. The following assertions are equivalent.

(i) $(ad_{f,B} \otimes (f \circ \lambda_H)) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B) = c_{B,B}^{-1} \circ ((f \circ \lambda_H) \otimes ad_{f,B}) \circ (\delta_H \otimes B)$.

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(ii) B is a left H-module comonoid via $ad_{f,B}$.

As a consequence, if λ_H is an isomorphism we have that H is a left H-module comonoid via ad_H if and only if (H, ad_H) is in the cocommutativity class of H.

Definition

A left-left entwining structure on $\mathcal{C}$ consists of a triple $(A, D, \psi_{A,D})$, where A is a monoid, D a comonoid, and $\psi_{A,D} : A \otimes D \to D \otimes A$ is a morphism satisfying the conditions

(a1) $\psi_{A,D} \circ (\eta_A \otimes D) = D \otimes \eta_A$,

(a2) $(D \otimes \mu_A) \circ (\psi_{A,D} \otimes A) \circ (A \otimes \psi_{A,D}) = \psi_{A,D} \circ (\mu_A \otimes D)$,

(a3) $(\delta_D \otimes A) \circ \psi_{A,D} = (D \otimes \psi_{A,D}) \circ (\psi_{A,D} \otimes D) \circ (A \otimes \delta_D)$,

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If we only have the conditions (a1) and (a2) we will say that $(A, D, \psi_{A,D})$ is a left-left semi-entwining structure.

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In a similar way, we can define the notions of right-right, right-left and left-right (semi)entwining structure.
For example, $(A, D, \psi_{D,A} : D \otimes A \to A \otimes D)$ will be a right-right semi-entwining structure if conditions

(b1) $\psi_{D,A} \circ (D \otimes \eta_A) = \eta_A \otimes D$,

(b2) $(\mu_A \otimes D) \circ (A \otimes \psi_{D,A}) \circ (\psi_{D,A} \otimes A) = \psi_{D,A} \circ (D \otimes \mu_A)$,

hold.

Definition

Let X and Y be monoids and comonoids and let $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ be a morphism. We will say that $\psi_{Y,X}$ is in the cocommutativity class of Y if the following equality

$$(\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X) = (c_{Y,X}^{-1} \otimes Y) \circ (Y \otimes \psi_{Y,X}) \circ (\delta_Y \otimes X),$$

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holds.

Lemma

Let X and Y be monoids and comonoids and let $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ be a morphism such $(\varepsilon_X \otimes Y) \circ \psi_{Y,X} = Y \otimes \varepsilon_X$ holds. The following assertions are equivalent.

(i) $\delta_{X \otimes Y} \circ \psi_{Y,X} = (\psi_{Y,X} \otimes \psi_{Y,X}) \circ \delta_{Y \otimes X}$.

(ii) $\psi_{Y,X}$ is in the cocommutativity class of Y, and satisfy the conditions

$$(\delta_X \otimes Y) \circ \psi_{Y,X} = (X \otimes \psi_{Y,X}) \circ (\psi_{Y,X} \otimes X) \circ (Y \otimes \delta_X),$$

$$(X \otimes \delta_Y) \circ \psi_{Y,X} = (\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X) \qquad (1)$$

Proposition

Let X and Y be bimonoids. The following assertions are equivalent.

(i) There is a morphism $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ such that $(Y, X, \psi_{Y,X})$ is a left-left entwining structure and $(X, Y, \psi_{Y,X})$ a right-right semi-entwining structure satisfying (1).

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Proof

(i) $\Rightarrow$ (ii) Define $\phi_X = (X \otimes \varepsilon_Y) \circ \psi_{Y,X}$.
(ii) $\Rightarrow$ (i) Define $\psi_{Y,X} = (\phi_X \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X)$.

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Proof

(i) $\Rightarrow$ (ii) Define $\phi_X = (X \otimes \varepsilon_Y) \circ \psi_{Y,X}$.
(ii) $\Rightarrow$ (i) Define $\psi_{Y,X} = (\phi_X \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X)$.

Moreover, $\psi_{Y,X}$ is in the cocommutativity class of Y iff so is (X, ϕ_X).

Definition

Let $\beta : X \to Y$ be a morphism of Hopf monoids and let $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ be a morphism. We will say that $\mathbf{X_Y} = (X, Y, \beta)$ is a *crossed module of Hopf monoids* if

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Equivalently, there is a morphism $\phi_X : Y \otimes X \to X$ such that

(d1) (X, ϕ_X) is a left Y-module monoid and comonoid.

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If λ_X is an isomorphism, $\mathbf{X_X} = (X, X, id_X)$ is a crossed module of Hopf monoids $\phi_X = ad_X$.

Definition

A morphism between two crossed modules of Hopf monoids $\mathbf{X_Y} = (X, Y, \beta)$ and $\mathbf{T_G} = (T, G, \partial)$ is a pair of Hopf monoid morphisms

$$u : X \to T, \quad v : Y \to G$$

such that

$$v \circ \beta = \partial \circ u, \quad (u \otimes \varepsilon_Y) \circ \psi_{Y,X} = (T \otimes \varepsilon_G) \circ \psi_{G,T} \circ (v \otimes u).$$

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$$v \circ \beta = \partial \circ u, \quad u \circ \phi_X = \phi_T \circ (v \otimes u).$$

Crossed products of crossed modules of Hopf monoids

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$$X\#Y = (X \otimes Y, \eta_{X\#Y} = \eta_X \otimes \eta_Y, \mu_{X\#Y} = (\mu_X \otimes \mu_Y) \circ (X \otimes \psi_{Y,X} \otimes Y)),$$

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Proposition

Let X and Y be Hopf monoids and let $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ be a morphism such that $(Y, X, \psi_{Y,X})$ is a left-left entwining structure and $(X, Y, \psi_{Y,X})$ a right-right semi-entwining structure such that $\psi_{Y,X}$ is in the cocommutativity class of Y and (1) holds. Then the tensor product comonoid structure is compatible with the smash product monoid structure, making

$$X \bowtie Y = (X \otimes Y, \eta_{X\bowtie Y} = \eta_{X\#Y}, \mu_{X\bowtie Y} = \mu_{X\#Y}, \varepsilon_{X\bowtie Y} = \varepsilon_X \otimes \varepsilon_Y, \delta_{X\bowtie Y} = \delta_{X\otimes Y})$$

a Hopf monoid with antipode $\lambda_{X\bowtie Y} = \psi_{Y,X} \circ (\lambda_Y \otimes \lambda_X) \circ c_{X,Y}$.

The main goal of this section is to construct the crossed product of two crossed modules of Hopf monoids. In order to do so, in what follows we consider two crossed modules of Hopf monoids $\mathbf{X_Y} = (X, Y, \beta)$ and $\mathbf{T_G} = (T, G, \partial)$ and denote the corresponding morphisms by $\psi_{Y,X}$ and $\psi_{G,T}$, respectively.

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Moreover, let $t : Y \otimes T \to X$ be a morphism and assume that

$$\psi_{G,X} : G \otimes X \to X \otimes G, \quad \psi_{T,X} : T \otimes X \to X \otimes T, \quad \psi_{G,Y} : G \otimes Y \to Y \otimes G$$

are three morphisms that induce left-left entwining structures and right-right semi-entwining structures and such that $\psi_{G,X}$ is in the class of cocommutativity of G, $\psi_{T,X}$ is in the class of cocommutativity of T, $\psi_{G,Y}$ is in the class of cocommutativity of G, (1) holds for the previous morphisms and the Yang-Baxter condition

$$(\psi_{Y,X} \otimes G) \circ (Y \otimes \psi_{G,X}) \circ (\psi_{G,Y} \otimes X) = (X \otimes \psi_{G,Y}) \circ (\psi_{G,X} \otimes Y) \circ (G \otimes \psi_{Y,X})$$

also holds.

Now define the morphism

$$\phi_{X \bowtie T} : Y \bowtie G \otimes X \bowtie T \to X \bowtie T$$

as

$$\phi_{X \bowtie T} =$$

$$(\mu_X \otimes T) \circ (X \otimes t \otimes T) \circ (X \otimes Y \otimes \delta_T \otimes \varepsilon_G) \circ (\psi_{Y,X} \otimes \psi_{G,T}) \circ (Y \otimes \psi_{G,X} \otimes T).$$

Lemma

The following assertions are equivalent.

(i) $(X \bowtie T, \phi_{X \bowtie T})$ is a left $Y \bowtie G$-module.

(ii) The equalities

$$t \circ (\eta_Y \otimes T) = \varepsilon_T \otimes \eta_X, \tag{2}$$

$$(t \otimes \varepsilon_G) \circ (Y \otimes \psi_{G,T}) \circ (\psi_{G,Y} \otimes T) = (X \otimes \varepsilon_G) \circ \psi_{G,X} \circ (G \otimes t), \tag{3}$$

and

$$t \circ (\mu_Y \otimes T) = \mu_X \circ (X \otimes t) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes t \otimes T) \circ (Y \otimes Y \otimes \delta_T) \tag{4}$$

hold.

Lemma

The following assertions are equivalent.

(i) $\phi_{X\bowtie T}$ is a monoid morphism.

(ii) The equalities

$$t\circ(Y\otimes\eta_T)=\varepsilon_Y\otimes\eta_X, \tag{5}$$

$$t\circ(Y\otimes\mu_T)= \tag{6}$$

$$(\mu_X\otimes\varepsilon_T)\circ(t\otimes\psi_{T,X})\circ(Y\otimes\delta_T\otimes X)\circ(Y\otimes T\otimes t)\circ(Y\otimes c_{Y,T}\otimes T)\circ(\delta_Y\otimes T\otimes T)$$

and

$$\mu_X\circ(X\otimes t)\circ(\psi_{Y,X}\otimes T)\circ(Y\otimes\psi_{T,X}) \tag{7}$$

$$=(\mu_X\otimes\varepsilon_T)\circ(t\otimes\psi_{T,X}\otimes\varepsilon_Y)\circ(Y\otimes\delta_T\otimes\psi_{Y,X})\circ(Y\otimes c_{Y,T}\otimes X)\circ(\delta_Y\otimes T\otimes X),$$

hold.

Lemma

The following assertions are equivalent.

(i) $\phi_{X\bowtie T}$ is a comonoid morphism.

(ii) t is a comonoid morphism and the equality

$$c_{X,T} \circ (t \otimes T) \circ (Y \otimes \delta_T) = (T \otimes t) \circ (c_{Y,T} \otimes T) \circ (Y \otimes \delta_T) \tag{8}$$

holds.

Lemma

If (2) and (5) hold,

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$$t \circ (\eta_Y \otimes T) = \varepsilon_T \otimes \eta_X, \quad t \circ (Y \otimes \eta_T) = \varepsilon_Y \otimes \eta_X$$

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(i) $(X \bowtie T, \phi_{X\bowtie T})$ is in the cocommutativity class of $Y \bowtie G$.

(ii) The equality

$$(t \otimes Y) \circ (Y \otimes c_{Y,T}) \circ (\delta_Y \otimes T) = c_{Y,X} \circ (Y \otimes t) \circ (\delta_Y \otimes T) \tag{9}$$

holds.

Lemma

The following assertions are equivalent.

(i) $(\beta \otimes \partial) \circ \phi_{X \bowtie T} = ad_{Y \bowtie G} \circ (Y \otimes G \otimes \beta \otimes \partial)$

(ii) The equalities

$$((\beta \circ t) \otimes \partial) \circ (Y \otimes \delta_T) = (\mu_Y \otimes G) \circ (Y \otimes (\psi_{G,Y} \circ c_{Y,G} \circ (\lambda_Y \otimes \partial))) \circ (\delta_Y \otimes T) \quad (10)$$

and

$$(\beta \otimes G) \circ \psi_{G,X} = \psi_{G,Y} \circ (G \otimes \beta) \quad (11)$$

hold.

Lemma (Peiffer identity)

The following assertions are equivalent.

(i) $\phi_{X\bowtie T}\circ(\beta\otimes\partial\otimes X\otimes T)=ad_{X\bowtie T}$

(ii) The equalities

$$(t\otimes T)\circ(\beta\otimes\delta_T)=(\mu_X\otimes T)\circ(X\otimes(\psi_{T,X}\circ c_{X,T}\circ(\lambda_X\otimes T)))\circ(\delta_X\otimes T) \quad (12)$$

and

$$\psi_{G,X}\circ(\partial\otimes X)=(X\otimes\partial)\circ\psi_{T,X} \quad (13)$$

hold.

Theorem

In the conditions of this section, the following assertions are equivalent.

(i) $\mathbf{X_Y} \bowtie \mathbf{T_G} = (X \bowtie T, Y \bowtie G, \beta \otimes \partial)$ is a crossed module of Hopf monoids via $\phi_{X \bowtie T}$.

(ii) t is a comonoid morphism and the equalities (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12) and (13) hold.

Projections

We assume that every idempotent morphism $q : Y \to Y$ in $\mathcal{C}$ splits, i.e., there exist an object Z (image of q) and morphisms $i : Z \to Y$ (injection) and $p : Y \to Z$ (projection) such that $q = i \circ p$ and $p \circ i = id_Z$.

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Definition

A *projection of Hopf monoids* is a quartet (T, B, u, w) where T, B are Hopf monoids, and $u : T \to B$, $w : B \to T$ are Hopf monoid morphisms such that $w \circ u = id_T$.
A morphism between projections of Hopf monoids (T, B, u, w) and (G, H, v, y) is a pair (∂, γ), where $\partial : T \to G$, $\gamma : B \to H$ are Hopf monoid morphisms such that

$$v \circ \partial = \gamma \circ u, \quad \partial \circ w = y \circ \gamma.$$

- Let (T, B, u, w) be a projection of Hopf monoids. The morphism

$$q_B = \mu_B \circ (B \otimes (u \circ \lambda_T \circ w)) \circ \delta_B$$

is an idempotent and, as a consequence, there exist an epimorphism p_B, a monomorphism i_B, and an object B^{coT} (submonoid of coinvariants) such that the diagram

$$\begin{tikzcd} B \arrow[rr, "q_B"] \arrow[dr, "p_B"'] & & B \\ & B^{coT} \arrow[ur, "i_B"'] & \end{tikzcd}$$

commutes and $p_B \circ i_B = id_{B^{coT}}$.

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commutes and $p_B \circ i_B = id_{B^{coT}}$.

- Also,

$$\begin{tikzcd} B^{coT} \arrow[r, "i_B"] & B \arrow[r, shift left, "(B\otimes w)\circ\delta_B"] \arrow[r, shift right, "B\otimes\eta_T"'] & B\otimes T \end{tikzcd}$$

is an equalizer diagram and

$$\begin{tikzcd} B\otimes T \arrow[r, shift left, "\mu_B\circ(B\otimes u)"] \arrow[r, shift right, "B\otimes\varepsilon_T"'] & B \arrow[r, "p_B"] & B^{coT} \end{tikzcd}$$

is a coequalizer diagram.

- The morphism i_B (p_B) is a monoid (comonoid) morphism, where the monoid and comonoid structures in B^{coT} are

$$\eta_{B^{coT}} = p_B \circ \eta_B, \quad \mu_{B^{coT}} = p_B \circ \mu_B \circ (i_B \otimes i_B),$$

$$\varepsilon_{B^{coT}} = \varepsilon_B \circ i_B, \quad \delta_{B^{coT}} = (p_B \otimes p_B) \circ \delta_B \circ i_B$$

respectively.

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  respectively.

- The morphism $ad_{u,B} \circ (T \otimes i_B)$ factorizes through the equalizer i_B, and the factorization

$$\varphi_{B^{coT}} = p_B \circ \mu_B \circ (u \otimes i_B) : T \otimes B^{coT} \to B^{coT}$$

  gives a left T-module monoid and comonoid structure for B^{coT}.

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  respectively.
- The morphism $ad_{u,B} \circ (T \otimes i_B)$ factorizes through the equalizer i_B, and the factorization

$$\varphi_{B^{coT}} = p_B \circ \mu_B \circ (u \otimes i_B) : T \otimes B^{coT} \to B^{coT}$$

  gives a left T-module monoid and comonoid structure for B^{coT}.
- If i_B is a comonoid morphism, B^{coT} is a Hopf monoid with antipode

$$\lambda_{B^{coT}} = p_B \circ \lambda_B \circ i_B.$$

- The morphism i_B (p_B) is a monoid (comonoid) morphism, where the monoid and comonoid structures in B^{coT} are

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- The morphism $ad_{u,B} \circ (T \otimes i_B)$ factorizes through the equalizer i_B, and the factorization

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  gives a left T-module monoid and comonoid structure for B^{coT}.
- If i_B is a comonoid morphism, B^{coT} is a Hopf monoid with antipode

$$\lambda_{B^{coT}} = p_B \circ \lambda_B \circ i_B.$$

- Finally, there is a Hopf monoid isomorphism between $B^{coT} \bowtie T$ and B defined as

$$\pi_B = \mu_B \circ (i_B \otimes u)$$

  and with inverse $\pi_B^{-1} = (p_B \otimes w) \circ \delta_B$.

Definition

Let $\mathbf{T_G} = (T, G, \partial)$ and $\mathbf{B_H} = (B, H, \gamma)$ be crossed modules of Hopf monoids and assume that (T, B, u, w) and (G, H, v, y) are projections of Hopf monoids. We say that

$$(\mathbf{T_G}, \mathbf{B_H}, (u, v), (w, y))$$

is a *projection of crossed modules of Hopf monoids* if (∂, γ) is a morphism between (T, B, u, w) and (G, H, v, y) such that the equalities

$$(u \otimes \varepsilon_G) \circ \psi_{G,T} = (B \otimes \varepsilon_H) \circ \psi_{H,B} \circ (v \otimes u),$$

$$(w \otimes \varepsilon_H) \circ \psi_{H,B} = (T \otimes \varepsilon_G) \circ \psi_{G,T} \circ (y \otimes w),$$

hold.

Definition

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hold.

Equivalently, if ϕ_T and ϕ_B are the left G-module and H-module monoid and comonoid structures for T and B, respectively, and the following equalities hold:

$$u \circ \phi_T = \phi_B \circ (v \otimes u), \quad w \circ \phi_B = \phi_T \circ (y \otimes w).$$

Theorem

Let $\mathbf{T_G} = (T, G, \partial)$ and $\mathbf{B_H} = (B, H, \gamma)$ be crossed modules of Hopf monoids. Let

$$(\mathbf{T_G}, \mathbf{B_H}, (u, v), (w, y))$$

be a projection of crossed modules of Hopf monoids such that i_B and i_H are comonoid morphisms. Then

$$\mathbf{B^{coT}}_{\mathbf{H^{coG}}} = (B^{coT}, H^{coG}, \sigma = p_H \circ \gamma \circ i_B)$$

is a crossed module of Hopf monoids where the left H^{coG}-module structure for B^{coT} is

$$\phi_{B^{coT}} = p_B \circ \phi_B \circ (i_H \otimes i_B).$$

Theorem

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is a crossed module of Hopf monoids and

$$\mathbf{B^{coT}}_{\mathbf{H^{coG}}} \bowtie \mathbf{T_G} \simeq \mathbf{B_H}$$

as crossed modules of Hopf monoids.

$$\mathbf{B^{coT}}_{\mathbf{H^{coG}}} \bowtie \mathbf{T_G} \simeq \mathbf{B_H}$$

Complete details in:

$$\mathbf{B^{coT}}{}_{\mathbf{H^{coG}}} \bowtie \mathbf{T_G} \simeq \mathbf{B_H}$$

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Alonso Álvarez, J.N., Fernández Vilaboa, J.M. y González Rodríguez, R. Crossed products of crossed modules of Hopf algebras, Theory and Applications of Categories 33, 867-897 (2018)

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Thank you