Iterated weak crossed products

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Abstract

In this paper we show how to iterate weak crossed products with common monoid. More concretely, if $(A \otimes V, \mu_{A\otimes V})$ and $(A \otimes W, \mu_{A\otimes W})$ are weak crossed products, we find sufficient conditions to obtain a new weak crossed product $(A \otimes V \otimes W, \mu_{A\otimes V\otimes W})$ that, in general, it is not linked with distributive laws.

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Introduction

Let A be a monoid and let V be an object living in a strict monoidal category $\mathcal{C}$ where every idempotent morphism splits. In [2] an associative product, called the weak crossed product of A and V, was defined on the tensor product $A\otimes V$ working with quadruples $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ where $\psi_V^A : V\otimes A \to A\otimes V$ and $\sigma_V^A : V \otimes V \to A \otimes V$ are morphisms satisfying some twisted-like and cocycle-like conditions. Associated to these morphisms we define an idempotent morphism $\nabla_{A\otimes V} : A \otimes V \to A \otimes V$ whose image, denoted by $A \times V$, inherits the associative product from $A \otimes V$. In order to define a unit for $A \times V$, and hence to obtain a monoid structure in this object, we complete the theory in [14] using the notion of preunit introduced by Caenepeel and De Groot in [10]. The theory presented in [2] and [14] contains, as particular instances, crossed products where $\nabla_{A\otimes V} = id_{A\otimes V}$, for example the one defined by Brzeziński in [9] or the notion of unified crossed product introduced by Agore and Militaru in [1], as well as crossed products where $\nabla_{A\otimes V} \neq id_{A\otimes V}$ like, for example, the weak smash product given by Caenepeel and De Groot in [10], the notion of weak wreath products that we can find in [27], the weak crossed products for weak bialgebras given in [25] (see also [14]) and, as was proved in [16], the partial crossed products introduced by Alves, Batista, Dokuchaev and Paques in [23]. Also, Böhm showed in [5] that a monad in the weak version of the Lack and Street's 2-category of monads in a 2-category is identical to a crossed product system in the sense of [2]. Finally, weak crossed products appears in a natural way in the study of bilinear factorizations of algebras [7], double crossed products of weak bialgebras [8], and weak projections of weak Hopf algebras [15].

The purpose of this paper is to find an alternative iteration process for weak crossed products with common monoid. Our main motivation comes from some interesting examples that can be found in the recent literature. For example, in [17], Jara, López, Panaite and Van Oystaeyen, motivated by the problem of defining a suitable representative for the product of spaces in noncommutative geometry, introduced the notion of iterated twisted tensor products of algebras. A good particular case of this iterated twisted tensor product can be found in [22], where Majid constructed an iterated sequence of double cross products of certain bialgebras. On the other hand, in [24], Panaite proved that under suitable conditions a Brzeziński crossed product may be iterated with a mirror version obtaining a new algebra structure. This construction contains as examples the iterated twisted tensor product of algebras and the quasi-Hopf two-sided smash product. Finally, in [12], Cheng developed the iteration process for wreath products and, on the other hand, using the 2-category of weak distributive laws, Böhm describe in [6] a method of iterating Street's weak wreath product construction (see [27]). Note that in the first examples of this paragraph the crossed products that we considered are cases where the associated idempotent is the identity and in the last one it is not the identity.

An outline of the paper is as follows. In the first section we resume the basic facts about weak crossed products proved in [14]. Also, in this section, if $\mathcal{K}$ is the one-object 2-category corresponding with a strict monoidal category $\mathcal{C}$, following [5], we describe in detail the 2-category $\mathsf{EM}^{w}(\mathcal{K})$ and the relation between monads in $\mathsf{EM}^{w}(\mathcal{K})$ and weak crossed products in $\mathcal{C}$. More concretely, in this setting the conclusion is the following: a monad in $\mathsf{EM}^{w}(\mathcal{K})$ is a weak crossed product with preunit in the category $\mathcal{C}$, and every weak crossed product with preunit can be interpreted as a monad in $\mathsf{EM}^{w}(\mathcal{K})$. Then, we can apply the general theory of composite monads and (weak) distributive laws (see [3], [12], [27]) to obtain iterations of weak crossed products. In Theorem 1.6 (Theorem 1.7), we give a concrete description of the (weak) distributive laws between monads in $\mathsf{EM}^{w}(\mathcal{K})$ and, as a corollary, we obtain that, if $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ and $\mathbb{A}_W = (A, W, \psi_W^A, \sigma_W^A)$ are quadruples satisfying the suitable conditions that permit to obtain two weak crossed products $(A\otimes V, \mu_{A\otimes V})$ and in $(A\otimes W, \mu_{A\otimes W})$, a (weak) distributive law of the monad induced by $(A\otimes V, \mu_{A\otimes V})$ in $\mathsf{EM}^{w}(\mathcal{K})$ over the corresponding monad induced by $(A\otimes W, \mu_{A\otimes W})$, is a morphism $\lambda : W\otimes V \to A\otimes V\otimes W$ satisfying some suitable conditions contained in Corollary 1.8 (Corollary 1.9). As a consequence, we obtain a iterated weak crossed product induced by λ that we called the λ-iteration of $(A\otimes V, \mu_{A\otimes V})$ and $(A\otimes W, \mu_{A\otimes W})$.

In section 3, we introduce a process to iterate weak crossed products not linked with distributive laws. Given two quadruples $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ and $\mathbb{A}_W = (A, W, \psi_W^A, \sigma_W^A)$, satisfying the twisted and cocycle conditions, and $(A\otimes V, \mu_{A\otimes V})$, $(A\otimes W, \mu_{A\otimes W})$ its associated weak crossed products, in this section we introduce the notions of link and twisting morphism between $\mathbb{A}_V$ and $\mathbb{A}_W$, proving that, if they exist, it is possible to construct a new quadruple $\mathbb{A}_{V\otimes W} = (A, V\otimes W, \psi_{V\otimes W}^A, \sigma_{V\otimes W}^A)$, satisfying the conditions that guarantee the existence of a new weak crossed product $(A\otimes V\otimes W, \mu_{A\otimes V\otimes W})$ called the iterated weak crossed product of $(A\otimes V, \mu_{A\otimes V})$ and $(A\otimes W, \mu_{A\otimes W})$. Also, if $(A\otimes V, \mu_{A\otimes V})$ and $(A\otimes W, \mu_{A\otimes W})$ admits a preunit, we find conditions to construct a preunit for $(A\otimes V\otimes W, \mu_{A\otimes V\otimes W})$. Finally, in Theorem 2.8, we prove that the associated monad in $\mathsf{EM}^{w}(\mathcal{K})$ for $(A\otimes V\otimes W, \mu_{A\otimes V\otimes W})$ is the canonical retract monad induced by an idempotent 2-cell in $\mathsf{EM}^{w}(\mathcal{K})$.

In the fourht section we discuss some examples involving wreath products, weak wreath products and the iteration process for Brzeziński crossed products proposed recently by Dăuş and Panaite in [13]. Finally, in the last section, following the results proved in [15] we obtain a new characterization of the iteration process proposed in section 3.

Throughout this paper $\mathcal{C}$ denotes a strict monoidal category with tensor product $\otimes$, unit object K. There is no loss of generality in assuming that $\mathcal{C}$ is strict because by Theorem XI.5.3 of [18] (this result implies the Mac Lane's coherence theorem) we know that every monoidal category is monoidally equivalent to a strict one. Then, we may work as if the constrains were all identities. We also assume that in $\mathcal{C}$ every idempotent morphism splits, i.e., for any morphism $q : M \to M$ such that $q\circ q = q$ there exists an object N, called the image of q, and morphisms $i : N \to M$, $p : M \to N$ such that $q = i\circ p$ and $p\circ i = id_N$. The morphisms p and i will be called a factorization of q. Note that Z, p and i are unique up to isomorphism. The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with (co)equalizers. Finally, given objects A, B, D and a morphism $f : B \to D$, we write $A\otimes f$ for $id_A\otimes f$ and $f\otimes A$ for $f\otimes id_A$.

An monoid in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A\otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A\circ(A\otimes\eta_A) = id_A = \mu_A\circ(\eta_A\otimes A)$, $\mu_A\circ(A\otimes\mu_A) = \mu_A\circ(\mu_A\otimes A)$. Given two monoids $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is a monoid morphism if $\mu_B\circ(f\otimes f) = f\circ\mu_A$, $f\circ\eta_A = \eta_B$.

A comonoid in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D\otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D\otimes D)\circ\delta_D = id_D = (D\otimes\varepsilon_D)\circ\delta_D$, $(\delta_D\otimes D)\circ\delta_D = (D\otimes\delta_D)\circ\delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \to E$ is a comonoid morphism if $(f\otimes f)\circ\delta_D = \delta_E\circ f$, $\varepsilon_E\circ f = \varepsilon_D$.

Let A be a monoid. The pair (M, φ_M) is a left A-module if M is an object in $\mathcal{C}$ and $\varphi_M : A\otimes M \to M$ is a morphism in $\mathcal{C}$ satisfying $\varphi_M \circ (\eta_A \otimes M) = id_M$, $\varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M)$. Given two left A-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left A-modules if $\varphi_N \circ (A\otimes f) = f \circ \varphi_M$. In a similar way we can define the notions of right A-module and morphism of right A-modules. In this case we denote the left action by ϕ_M.

1. Weak crossed products

In the first paragraphs of this section we resume some basic facts about the general theory of weak crossed products. The complete details can be found in [14].

Let A be a monoid and V be an object in $\mathcal{C}$. Suppose that there exists a morphism

$$\psi_V^A : V \otimes A \to A \otimes V$$

such that the following equality holds

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\psi_V^A \otimes A) = \psi_V^A \circ (V \otimes \mu_A). \tag{1}$$

As a consequence of (1), the morphism $\nabla_{A\otimes V} : A \otimes V \to A \otimes V$ defined by

$$\nabla_{A\otimes V} = (\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (A \otimes V \otimes \eta_A) \tag{2}$$

is idempotent. Moreover, $\nabla_{A\otimes V}$ satisfies that

$$\nabla_{A\otimes V} \circ (\mu_A \otimes V) = (\mu_A \otimes V) \circ (A \otimes \nabla_{A\otimes V}),$$

that is, $\nabla_{A\otimes V}$ is a left A-module morphism (see Lemma 3.1 of [14]) for the regular action $\varphi_{A\otimes V} = \mu_A \otimes V$. With $A \times V$, $i_{A\otimes V} : A\times V \to A \otimes V$ and $p_{A\otimes V} : A \otimes V \to A \times V$ we denote the object, the injection and the projection associated to the factorization of $\nabla_{A\otimes V}$. Finally, if ψ_V^A satisfies (1), the following identities hold

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\nabla_{A\otimes V} \otimes A) = (\mu_A \otimes V) \circ (A \otimes \psi_V^A) = \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \psi_V^A). \tag{3}$$

From now on we consider quadruples $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ where A is a monoid, V an object, $\psi_V^A : V \otimes A \to A \otimes V$ a morphism satisfiying (1) and $\sigma_V^A : V \otimes V \to A \otimes V$ a morphism in $\mathcal{C}$.

We say that $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ satisfies the twisted condition if

$$(\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \psi_V^A) \tag{4}$$

and the cocycle condition holds if

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\sigma_V^A \otimes V) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \sigma_V^A). \tag{5}$$

Note that, if $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ satisfies the twisted condition in Proposition 3.4 of [14] we prove that the following equalities hold:

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \nabla_{A\otimes V}) = \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V), \tag{6}$$

$$\nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nabla_{A\otimes V} \otimes V) = \nabla_{A\otimes V} \circ (\mu_A \otimes V) \circ (A \otimes \sigma_V^A). \tag{7}$$

Then, if $\nabla_{A\otimes V} \circ \sigma_V^A = \sigma_V^A$ we obtain

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V) \circ (V \otimes \nabla_{A\otimes V}) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\psi_V^A \otimes V), \tag{8}$$

$$(\mu_A \otimes V) \circ (A \otimes \sigma_V^A) \circ (\nabla_{A\otimes V} \otimes V) = (\mu_A \otimes V) \circ (A \otimes \sigma_V^A). \tag{9}$$

By virtue of (4) and (5) we will consider from now on, and without loss of generality, that

$$\nabla_{A\otimes V} \circ \sigma_V^A = \sigma_V^A \tag{10}$$

holds for all quadruples $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ (see Proposition 3.7 of [14]).

For $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$ define the product

$$\mu_{A\otimes V} = (\mu_A \otimes V) \circ (\mu_A \otimes \sigma_V^A) \circ (A \otimes \psi_V^A \otimes V) \tag{11}$$

and let $\mu_{A\times V}$ be the product

$$\mu_{A\times V} = p_{A\otimes V} \circ \mu_{A\otimes V} \circ (i_{A\otimes V} \otimes i_{A\otimes V}). \tag{12}$$

If the twisted and the cocycle conditions hold, the product $\mu_{A\otimes V}$ is associative and normalized with respect to $\nabla_{A\otimes V}$ (i.e. $\nabla_{A\otimes V}\circ\mu_{A\otimes V}=\mu_{A\otimes V}=\mu_{A\otimes V}\circ(\nabla_{A\otimes V}\otimes\nabla_{A\otimes V})$) and by the definition of $\mu_{A\otimes V}$ we have

$$\mu_{A\otimes V}\circ(\nabla_{A\otimes V}\otimes A\otimes V)=\mu_{A\otimes V} \tag{13}$$

and therefore

$$\mu_{A\otimes V}\circ(A\otimes V\otimes\nabla_{A\otimes V})=\mu_{A\otimes V}. \tag{14}$$

Due to the normality condition, $\mu_{A\times V}$ is associative as well (Propostion 3.8 of [14]). Hence we define:

Definition 1.1. If $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$ satisfies (4) and (5) we say that $(A\otimes V,\mu_{A\otimes V})$ is a weak crossed product.

The next natural question that arises is if it is possible to endow $A\times V$ with a unit, and hence with a monoid structure. As $A\times V$ is given as an image of an idempotent, it seems reasonable to use the notion of preunit introduced in [10] to obtain an unit. In our setting, if A is a monoid, V an object in $\mathcal{C}$ and $m_{A\otimes V}$ is an associative product defined in $A\otimes V$ a preunit $\nu_V:K\to A\otimes V$ is a morphism satisfying

$$m_{A\otimes V}\circ(A\otimes V\otimes\nu_V)=m_{A\otimes V}\circ(\nu_V\otimes A\otimes V),\ \ \nu_V=m_{A\otimes V}\circ(\nu_V\otimes\nu_V). \tag{15}$$

Associated to a preunit we obtain an idempotent morphism

$$\nabla_{A\otimes V}^{\nu_V}=m_{A\otimes V}\circ(A\otimes V\otimes\nu_V):A\otimes V\to A\otimes V.$$

Take $A\times V$ the image of this idempotent, $p_{A\otimes V}^{\nu_V}$ the projection and $i_{A\otimes V}^{\nu_V}$ the injection. It is possible to endow $A\times V$ with a monoid structure whose product is

$$m_{A\times V}=p_{A\otimes V}^{\nu_V}\circ m_{A\otimes V}\circ(i_{A\otimes V}^{\nu_V}\otimes i_{A\otimes V}^{\nu_V})$$

and whose unit is $\eta_{A\times V}=p_{A\otimes V}^{\nu_V}\circ\nu_V$ (see Proposition 2.5 of [14]). If moreover, $m_{A\otimes V}$ is left A-linear for the actions $\varphi_{A\otimes V}=\mu_A\otimes V$, $\varphi_{A\otimes V\otimes A\otimes V}=\varphi_{A\otimes V}\otimes A\otimes V$ and normalized with respect to $\nabla_{A\otimes V}^{\nu_V}$, the morphism

$$\beta_{\nu_V}:A\to A\otimes V,\ \beta_{\nu_V}=(\mu_A\otimes V)\circ(A\otimes\nu_V) \tag{16}$$

is multiplicative and left A-linear for $\varphi_A=\mu_A$.

Although β_{ν_V} is not a monoid morphism, because $A\otimes V$ is not a monoid, we have that $\beta_{\nu_V}\circ\eta_A=\nu_V$, and thus the morphism $\bar{\beta_{\nu_V}}=p_{A\otimes V}^{\nu}\circ\beta_{\nu_V}:A\to A\times V$ is a monoid morphism.

In light of the considerations made in the last paragraphs, and using the twisted and the cocycle conditions, in [14] we characterize weak crossed products with a preunit, and moreover we obtain a monoid structure on $A\times V$. These assertions are a consequence of the following results proved in [14].

Theorem 1.2. *Let A be a monoid, V an object and $m_{A\otimes V}:A\otimes V\otimes A\otimes V\to A\otimes V$ a morphism of left A-modules for the actions $\varphi_{A\otimes V}=\mu_A\otimes V$, $\varphi_{A\otimes V\otimes A\otimes V}=\varphi_{A\otimes V}\otimes A\otimes V$.*

Then the following statements are equivalent:

(i) *The product $m_{A\otimes V}$ is associative with preunit ν and normalized with respect to $\nabla_{A\otimes V}^{\nu_V}$.*
(ii) *There exist morphisms $\psi_V^A:V\otimes A\to A\otimes V$, $\sigma_V^A:V\otimes V\to A\otimes V$ and $\nu_V:k\to A\otimes V$ such that if $\mu_{A\otimes V}$ is the product defined in (11), the pair $(A\otimes V,\mu_{A\otimes V})$ is a weak crossed product with $m_{A\otimes V}=\mu_{A\otimes V}$ satisfying:*

$$(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\nu_V)=\nabla_{A\otimes V}\circ(\eta_A\otimes V), \tag{17}$$

$$(\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\nu_V\otimes V)=\nabla_{A\otimes V}\circ(\eta_A\otimes V), \tag{18}$$

$$(\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\nu_V\otimes A)=\beta_{\nu_V}, \tag{19}$$

where β_{ν_V} is the morphism defined in (16). In this case ν is a preunit for $\mu_{A\otimes V}$, the idempotent morphism of the weak crossed product $\nabla_{A\otimes V}$ is the idempotent $\nabla_{A\otimes V}^{\nu_V}$, and we say that the pair $(A\otimes V,\mu_{A\otimes V})$ is a weak crossed product with preunit ν_V.

Remark 1.3. Note that in the proof of the previous Theorem for $(i) \Rightarrow (ii)$ we define ψ_V^A and σ_V^A as

$$\psi_V^A = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \beta_{\nu_V}), \tag{20}$$

$$\sigma_V^A = m_{A\otimes V} \circ (\eta_A \otimes V \otimes \eta_A \otimes V). \tag{21}$$

Also, by (19), we have

$$\nabla_{A\otimes V} \circ \nu_V = \nu_V. \tag{22}$$

Corollary 1.4. *If $(A \otimes V, \mu_{A\otimes V})$ is a weak crossed product with preunit ν_V, then $A \times V$ is a monoid with the product defined in (12) and unit $\eta_{A\times V} = p_{A\otimes V} \circ \nu_V$.*

Let $\mathcal{K}$ be a 2-category. In [5] Böhm introduced the 2-category $\mathsf{EM}^w(\mathcal{K})$ as the weak version of Lack and Street's 2-category of monads in the 2-category $\mathcal{K}$ (see [19]). In the particular case of the one-object 2-category corresponding to $\mathcal{C}$ (i.e. the 2-category whose 0-cell is K, whose 1-cells are objects of $\mathcal{C}$, whose 2-cells are the morphisms of $\mathcal{C}$, whose horizontal composition is the tensor product of $\mathcal{C}$, and whose vertical composition is the composition of $\mathcal{C}$), $\mathsf{EM}^w(\mathcal{K})$ consists of:

- 0-cells are monoids S in $\mathcal{C}$.
- 1-cells $S \to T$ are pairs $(F, \psi_F^{S,T})$ consisting of an object F in $\mathcal{C}$ and a morphism $\psi_F^{S,T} : F \otimes T \to S \otimes F$ in $\mathcal{C}$ such that

$$\psi_F^{S,T} \circ (F \otimes \mu_T) = (\mu_S \otimes F) \circ (S \otimes \psi_F^{S,T}) \circ (\psi_F^{S,T} \otimes T). \tag{23}$$

  If $S = T$, the 1-cell $(F, \psi_F^{S,S})$ will be denoted by (F, ψ_F^S). The composition of 1-cells $(F, \psi_F^{S,T})$ and $(F', \psi_{F'}^{T,D})$, is defined by

$$(F', \psi_{F'}^{T,D}) \circ (F, \psi_F^{S,T}) = (F \otimes F', \psi_{F\otimes F'}^{S,D} = (\psi_F^{S,T} \otimes F') \circ (F \otimes \psi_{F'}^{T,D})).$$

  The identity cell is (K, id_S).
- 2-cells $(F, \psi_F^{S,T}) \Rightarrow (G, \psi_G^{S,T})$ are morphisms in $\mathcal{C}$, $\rho : F \to S \otimes G$, such that

$$(\mu_S \otimes G) \circ (S \otimes \rho) \circ \psi_F^{S,T} = (\mu_S \otimes G) \circ (S \otimes \psi_G^{S,T}) \circ (\rho \otimes T), \tag{24}$$

$$\rho = (\mu_S \otimes G) \circ (S \otimes \psi_G^{S,T}) \circ (\rho \otimes \eta_T). \tag{25}$$

  The identity 2-cell is $id_{(F,\psi_F^{S,T})} = \psi_F^{S,T} \circ (F \otimes \eta_T) : (F, \psi_F^{S,T}) \Rightarrow (F, \psi_F^{S,T})$.
  If

$$\rho : (F, \psi_F^{S,T}) \Rightarrow (G, \psi_G^{S,T}), \;\; \rho' : (F', \psi_{F'}^{T,D}) \Rightarrow (G', \psi_{G'}^{T,D})$$

  are 2-cells, the horizontal composition

$$\rho' \circledast \rho : (F', \psi_{F'}^{T,D}) \circ (F, \psi_F^{S,T}) = (F \otimes F', \psi_{F\otimes F'}^{S,D}) \Longrightarrow (G \otimes G', \psi_{G\otimes G'}^{S,D}) = (G', \psi_{G'}^{T,D}) \circ (G, \psi_G^{S,T})$$

  is defined by

$$\rho' \circledast \rho = (\mu_S \otimes G \otimes G') \circ (S \otimes \rho \otimes G') \circ (\psi_F^{S,T} \otimes G') \circ (F \otimes \rho') \tag{26}$$

Finally, the vertical composition of 2-cells $\rho : (F, \psi_F^{S,T}) \Rightarrow (G, \psi_G^{S,T})$, $\tau : (G, \psi_G^{S,T}) \Rightarrow (U, \psi_U^{S,T})$ is defined by

$$\tau \bullet \rho = (\mu_S \otimes U) \circ (S \otimes \tau) \circ \rho. \tag{27}$$

Following Section 2 of [5], a monad in $\mathsf{EM}^w(\mathcal{K})$ is giving by a triple $((F, \psi_F^S), \sigma_F^S, \nu_F)$, consisting of a 1-cell $(F, \psi_F^S) : S \to S$, and 2-cells $\sigma_F^S : (F, \psi_F^S) \circ (F, \psi_F^S) \Rightarrow (F, \psi_F^S)$, and $\nu_F : (K, id_S) \Rightarrow (F, \psi_F^S)$ in $\mathsf{EM}^w(\mathcal{K})$ such that

$$\sigma_F^S \bullet (\sigma_F^S \circledast id_{(F,\psi_F^S)}) = \sigma_F^S \bullet (id_{(F,\psi_F^S)} \circledast \sigma_F^S),$$

$$\sigma_F^S \bullet (\nu_F \circledast id_{(F,\psi_F^S)}) = id_{(F,\psi_F^S)} = \sigma_F^S \bullet (id_{(F,\psi_F^S)} \circledast \nu_F).$$

Then, by Theorem 1.1 of [5], this means an object F, and morphisms $\psi_F^S : F \otimes S \to S \otimes F$, $\sigma_F^S : F \otimes F \to S \otimes F$ and $\nu_F : K \to S \otimes F$, in $\mathcal{K}$, subject to the following identities:

$$\psi_F^S \circ (F \otimes \mu_S) = (\mu_S \otimes F) \circ (S \otimes \psi_F^S) \circ (\psi_F^S \otimes S), \tag{28}$$

$$(\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\psi_F^S \otimes F) \circ (F \otimes \psi_F^S) = (\mu_S \otimes F) \circ (S \otimes \psi_F^S) \circ (\sigma_F^S \otimes S), \tag{29}$$

$$(\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\sigma_F^S \otimes F) = (\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\psi_F^S \otimes F) \circ (F \otimes \sigma_F^S), \tag{30}$$

$$\sigma_F^S = (\mu_S \otimes F) \circ (S \otimes \psi_F^S) \circ (\sigma_F^S \otimes \eta_S), \tag{31}$$

$$(\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\psi_F^S \otimes F) \circ (F \otimes \nu_F) = \psi_F^S \circ (F \otimes \eta_S), \tag{32}$$

$$(\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\nu_F \otimes F) = \psi_F^S \circ (F \otimes \eta_S), \tag{33}$$

$$(\mu_S \otimes F) \circ (S \otimes \psi_F^S) \circ (\nu_F \otimes S) = (\mu_S \otimes F) \circ (S \otimes \nu_F). \tag{34}$$

As a consequence, by Theorem 1.2, we have that a monad in $\mathsf{EM}^w(\mathcal{K})$ is a weak crossed product associated to the quadruple $(S, F, \psi_F^S, \sigma_F^S)$ with preunit ν_F.

By Definition 2.1 of [5] a premonad in $\mathsf{EM}^w(\mathcal{K})$ is giving by a triple $((F, \psi_F^S), \sigma_F^S, u_F)$, consisting of a 1-cell $(F, \psi_F^S) : S \to S$, and 2-cells $\sigma_F^S : (F, \psi_F^S) \circ (F, \psi_F^S) \Rightarrow (F, \psi_F^S)$, and $\nu_F : (K, id_S) \Rightarrow (F, \psi_F^S)$ in $\mathsf{EM}^w(\mathcal{K})$ such that

$$\sigma_F^S \bullet (\sigma_F^S \circledast id_{(F,\psi_F^S)}) = \sigma_F^S \bullet (id_{(F,\psi_F^S)} \circledast \sigma_F^S),$$

$$\sigma_F^S \bullet (\nu_F \circledast id_{(F,\psi_F^S)}) = \sigma_F^S \bullet (id_{(F,\psi_F^S)} \circledast \nu_F),$$

$$\sigma_F^S \bullet (\nu_F \circledast \nu_F) = \nu_F,$$

$$\sigma_F^S \bullet (\sigma_F^S \circledast id_{(F,\psi_F^S)}) \bullet (\nu_F \circledast id_{(F\otimes F,\psi_{F\otimes F}^S)}) = \sigma_F^S.$$

Then, this means an object F, and morphisms $\psi_F^S : F \otimes S \to S \otimes F$, $\sigma_F^S : F \otimes F \to S \otimes F$ and $\nu_F : K \to S \otimes F$, in $\mathcal{K}$, subject to the identities (28), (29), (30), (31), (34) and

$$(\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\psi_F^S \otimes F) \circ (F \otimes \nu_F) = (\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\nu_F \otimes F), \tag{35}$$

$$(\mu_S \otimes F) \circ (S \otimes ((\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\nu_F \otimes F))) \circ \nu_F = \nu_F, \tag{36}$$

$$(\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (((\mu_S \otimes F) \circ (S \otimes \sigma_F^S) \circ (\nu_F \otimes F)) \otimes F) = \sigma_F^S. \tag{37}$$

Definition 1.5. Let $((F, \psi_F^S), \sigma_F^S, \nu_F)$, $((G, \psi_G^S), \sigma_G^S, \nu_G)$ be monads in $\mathsf{EM}^w(\mathcal{K})$. A distributive law of the monad $((F, \psi_F^S), \sigma_F^S, \nu_F)$ over the monad $((G, \psi_G^S), \sigma_G^S, \nu_G)$ is a 2-cell

$$\lambda : (F, \psi_F^S) \circ (G, \psi_G^S) \Rightarrow (G, \psi_G^S) \circ (F, \psi_F^S)$$

in $\mathsf{EM}^w(\mathcal{K})$ such that

$$\lambda \bullet (id_{(F,\psi_F^S)} \circledast \sigma_G^S) = (\sigma_G^S \circledast id_{(F,\psi_F^S)}) \bullet (id_{(G,\psi_G^S)} \circledast \lambda) \bullet (\lambda \circledast id_{(G,\psi_G^S)}), \tag{38}$$

$$\lambda \bullet (\sigma_F^S \circledast id_{(G,\psi_G^S)}) = (id_{(G,\psi_G^S)} \circledast \sigma_F^S) \bullet (\lambda \circledast id_{(F,\psi_F^S)}) \bullet (id_{(F,\psi_F^S)} \circledast \lambda), \tag{39}$$

$$\lambda \bullet (\nu_F \circledast id_{(G,\psi_G^S)}) = id_{(G,\psi_G^S)} \circledast \nu_F, \tag{40}$$

$$\lambda \bullet (id_{(F,\psi_F^S)} \circledast \nu_G) = \nu_G \circledast id_{(F,\psi_F^S)}. \tag{41}$$

This notion is a 2-categorical version of the one introduced by Beck in [4] (see Example 3.1). Following [27], a weak distributive law of the monad $((F, \psi_F^S), \sigma_F^S, \nu_F)$ over the monad $((G, \psi_G^S), \sigma_G^S, \nu_G)$ is a 2-cell

$$\lambda : (F, \psi_F^S) \circ (G, \psi_G^S) \Rightarrow (G, \psi_G^S) \circ (F, \psi_F^S)$$

in $\mathsf{EM}^w(\mathcal{K})$ satifying (38), (39), and

$$\lambda \bullet (\nu_F \circledast id_{(G,\psi_G^S)}) = (\sigma_G^S \circledast id_{(F,\psi_F^S)}) \bullet (id_{(G,\psi_G^S)} \circledast (\lambda \bullet (\nu_F \circledast \nu_G))), \tag{42}$$

$$\lambda \bullet (id_{(F,\psi_F^S)} \circledast \nu_G) = (id_{(G,\psi_G^S)} \circledast \sigma_F^S) \bullet ((\lambda \bullet (\nu_F \circledast \nu_G)) \circledast id_{(F,\psi_F^S)}). \tag{43}$$

Theorem 1.6. *Let $((F,\psi_F^S),\sigma_F^S,\nu_F)$, $((G,\psi_G^S),\sigma_G^S,\nu_G)$ be monads in $\mathsf{EM}^w(\mathcal{K})$. A distributive law of the monad $((F,\psi_F^S),\sigma_F^S,\nu_F)$ over the monad $((G,\psi_G^S),\sigma_G^S,\nu_G)$ is determined by a morphism*

$$\lambda : G\otimes F\to S\otimes F\otimes G$$

in $\mathcal{C}$ such that

$$(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\psi_G^S\otimes F)\circ(G\otimes\psi_F^S)=(\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(S\otimes F\otimes\psi_G^S)\circ(\lambda\otimes S),\quad (44)$$

$$\lambda=(\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(S\otimes F\otimes\psi_G^S)\circ(\lambda\otimes\eta_S),\quad (45)$$

$$(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\sigma_G^S\otimes F)\quad (46)$$
$$=(\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(\mu_S\otimes F\otimes\sigma_G^S)\circ(S\otimes\lambda\otimes G)\circ(\psi_G^S\otimes F\otimes G)\circ(G\otimes\lambda),$$

$$(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\psi_G^S\otimes F)\circ(G\otimes\sigma_F^S)\quad (47)$$
$$=(\mu_S\otimes F\otimes G)\circ(\mu_S\otimes\sigma_F^S\otimes G)\circ(S\otimes\psi_F^S\otimes F\otimes G)\circ(S\otimes F\otimes\lambda)\circ(\lambda\otimes F),$$

$$(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\psi_G^S\otimes F)\circ(G\otimes\nu_F)=(\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(\nu_F\otimes(\psi_G^S\circ(G\otimes\eta_S))),\quad (48)$$

$$(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\nu_G\otimes F)=(\psi_F^S\otimes G)\circ(F\otimes\nu_G).\quad (49)$$

Proof. By definition, a distributive law of the monad $((F,\psi_F^S),\sigma_F^S,\nu_F)$ over the monad $((G,\psi_G^S),\sigma_G^S,\nu_G)$ is a 2-cell,

$$\lambda : (G\otimes F,\psi_{G\otimes F}^S)\Rightarrow(F\otimes G,\psi_{F\otimes G}^S)$$

in $\mathsf{EM}^w(\mathcal{K})$. Then, $\lambda : G\otimes F\Rightarrow S\otimes F\otimes G$ is a morphism in $\mathcal{C}$ such that (24) and (25) hold and, equivalently, (44) and (45) hold.

On the other hand, note that

$$\begin{aligned}&\lambda\bullet(id_{(F,\psi_F^S)}\circledast\sigma_G^S)\\
&=(\mu_S\otimes F\otimes G)\circ(\mu_S\otimes\lambda)\circ(S\otimes\psi_G^S\otimes F)\circ(\sigma_G^S\otimes(\psi_F^S\circ(F\otimes\eta_S)))\\
&=(\mu_S\otimes F\otimes G)\circ(S\otimes((\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\psi_G^S\otimes F)\circ(G\otimes(\psi_F^S\circ(F\otimes\eta_S)))))\circ(\sigma_G^S\otimes F)\\
&=(\mu_S\otimes F\otimes G)\circ(S\otimes((\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(S\otimes F\otimes\psi_G^S)\circ(\lambda\otimes\eta_S)))\circ(\sigma_G^S\otimes F)\\
&=(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\sigma_G^S\otimes F)\end{aligned}$$

where the first equality follows by (28) for G, the second one follows by the associativity of μ_S, the third one is a consequence of (44), and the last one relies on (45).

Also, by (28) for G and F, by monad structure of S, (44), (29) and (31) for G, we obtain

$$\begin{aligned}&(\sigma_G^S\circledast id_{(F,\psi_F^S)})\bullet(id_{(G,\psi_G^S)}\circledast\lambda)\bullet(\lambda\circledast id_{(G,\psi_G^S)})\\
&=(\mu_S\otimes F\otimes G)\circ(S\otimes((\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(S\otimes F\otimes\sigma_G^S)\circ(((\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\\
&\quad\circ(\psi_G^S\otimes F)\circ(G\otimes\psi_F^S))\otimes G)))\circ(\psi_G^S\otimes F\otimes\psi_G^S)\circ(G\otimes\lambda\otimes\eta_S)\\
&=(\mu_S\otimes F\otimes G)\circ(\mu_S\otimes\psi_F^S\otimes G)\circ(S\otimes S\otimes F\otimes((\mu_S\otimes G)\circ(S\otimes\sigma_G^S)\circ(\psi_G^S\otimes G)\circ(G\otimes\psi_G^S)))\\
&\quad\circ(S\otimes\lambda\otimes G\otimes S)\circ(\psi_G^S\otimes F\otimes G\otimes S)\circ(G\otimes\lambda\otimes\eta_S)\\
&=(\mu_S\otimes F\otimes G)\circ(\mu_S\otimes\psi_F^S\otimes G)\circ(S\otimes S\otimes F\otimes((\mu_S\otimes G)\circ(S\otimes\psi_G^S)\circ(\sigma_G^S\otimes\eta_S)))\circ(S\otimes\lambda\otimes G)\\
&\quad\circ(\psi_G^S\otimes F\otimes G)\circ(G\otimes\lambda)\\
&=(\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(\mu_S\otimes F\otimes\sigma_G^S)\circ(S\otimes\lambda\otimes G)\circ(\psi_G^S\otimes F\otimes G)\circ(G\otimes\lambda)\end{aligned}$$

Therefore, (46) holds because (38) holds. Similarly, we obtain that

$$\lambda\bullet(\sigma_F^S\circledast id_{(G,\psi_G^S)})=(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\psi_G^S\otimes F)\circ(G\otimes\sigma_F^S)$$

and

$$\begin{aligned}&(id_{(G,\psi_G^S)}\circledast\sigma_F^S)\bullet(\lambda\circledast id_{(F,\psi_F^S)})\bullet(id_{(F,\psi_F^S)}\circledast\lambda)\\
&=(\mu_S\otimes F\otimes G)\circ(\mu_S\otimes\sigma_F^S\otimes G)\circ(S\otimes\psi_F^S\otimes F\otimes G)\circ(S\otimes F\otimes\lambda)\circ(\lambda\otimes F).\end{aligned}$$

Then, (47) holds.

The proof for (48) is the following: on the one hand, by (28) for G, we have

$$\begin{aligned}\lambda\bullet(\nu_F\circledast id_{(G,\psi_G^S)})&=(\mu_S\otimes F\otimes G)\circ(\mu_S\circledast\lambda)\circ(S\otimes(\psi_G^S\otimes(G\otimes\eta_S))\otimes F)\circ(\psi_G^S\otimes F)\circ(G\otimes\nu_F)\\
&=(\mu_S\otimes F\otimes G)\circ(S\otimes\lambda)\circ(\psi_G^S\otimes F)\circ(G\otimes\nu_F),\end{aligned}$$

and, on the other hand, by (34),

$$id_{(G,\psi_G^S)} \circledast \nu_F = (\mu_S \otimes F \otimes G) \circ (S \otimes \psi_F^S \otimes G) \circ (\nu_F \otimes (\psi_G^S \circ (G \otimes \eta_S))).$$

Therefore, (48) holds. Finally, the proof for (49) is similar and we leave the details to the reader.

□

In a similar way we can obtain

Theorem 1.7. *Let $((F, \psi_F^S), \sigma_F^S, \nu_F)$, $((G, \psi_G^S), \sigma_G^S, \nu_G)$ be monads in $\mathsf{EM}^w(\mathcal{K})$. A weak distributive law of the monad $((F, \psi_F^S), \sigma_F^S, \nu_F)$ over the monad $((G, \psi_G^S), \sigma_G^S, \nu_G)$ is determined by a morphism*

$$\lambda : G \otimes F \to S \otimes F \otimes G$$

in $\mathcal{C}$ satisfying (44), (45), (46), (47) and

$$(\mu_S \otimes F \otimes G) \circ (\mu_S \otimes \psi_F^S \otimes G) \circ (S \otimes S \otimes F \otimes \sigma_G^S) \circ (S \otimes (\gamma_S \circ \nu_F) \otimes G) \circ \psi_G^S \circ (G \otimes \eta_S), \tag{50}$$

$$= (\mu_S \otimes F \otimes G) \circ (S \otimes \lambda) \circ (\psi_G^S \otimes F) \circ (G \otimes \nu_F)$$

$$(\mu_S \otimes F \otimes G) \circ (\mu_S \otimes \psi_F^S \otimes G) \circ (S \otimes \sigma_F^S \otimes \psi_G^S) \circ (\psi_F^S \otimes F \otimes G \otimes \eta_S) \circ (F \otimes (\gamma_S \circ \nu_F)) = \gamma_S \circ \psi_F^S \circ (F \otimes \eta_S), \tag{51}$$

where $\gamma_S = (\mu_S \otimes F \otimes G) \circ (\mu_S \otimes \lambda) \circ (S \otimes \nu_G \otimes F)$.

As a consequence of the previous theorems we have the following results.

Corollary 1.8. *Let $(A \otimes V, \mu_{A\otimes V})$ and $(A \otimes W, \mu_{A\otimes W})$ be weak crossed products with preunits ν_V and ν_W. A distributive law of $(A \otimes V, \mu_{A\otimes V})$ over $(A \otimes W, \mu_{A\otimes W})$, i.e. a distributive law of the monad induced by $(A \otimes V, \mu_{A\otimes V})$ in $\mathsf{EM}^w(\mathcal{K})$ over the corresponding monad induced by $(A \otimes W, \mu_{A\otimes W})$, is a morphism*

$$\lambda : W \otimes V \to A \otimes V \otimes W$$

in $\mathcal{C}$ such that

$$(\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\psi_W^A \otimes V) \circ (W \otimes \psi_V^A) = (\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (A \otimes V \otimes \psi_W^A) \circ (\lambda \otimes A), \tag{52}$$

$$\lambda = (\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (A \otimes V \otimes \psi_W^A) \circ (\lambda \otimes \eta_A), \tag{53}$$

$$(\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\sigma_W^A \otimes V) \tag{54}$$

$$= (\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (\mu_A \otimes V \otimes \sigma_W^A) \circ (A \otimes \lambda \otimes W) \circ (\psi_W^A \otimes V \otimes W) \circ (W \otimes \lambda),$$

$$(\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\psi_W^A \otimes V) \circ (W \otimes \sigma_V^A) \tag{55}$$

$$= (\mu_A \otimes V \otimes W) \circ (\mu_A \otimes \sigma_V^A \otimes W) \circ (A \otimes \psi_V^A \otimes V \otimes W) \circ (A \otimes V \otimes \lambda) \circ (\lambda \otimes V),$$

$$(\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\psi_W^A \otimes V) \circ (W \otimes \nu_V) = (\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (\nu_V \otimes (\psi_W^A \circ (W \otimes \eta_A))), \tag{56}$$

$$(\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\nu_W \otimes V) = (\psi_V^A \otimes W) \circ (V \otimes \nu_W). \tag{57}$$

Corollary 1.9. *Let $(A \otimes V, \mu_{A\otimes V})$ and $(A \otimes W, \mu_{A\otimes W})$ be weak crossed products with preunits ν_V and ν_W. A weak distributive law of $(A \otimes V, \mu_{A\otimes V})$ over $(A \otimes W, \mu_{A\otimes W})$, i.e. a weak distributive law of the monad induced by $(A \otimes V, \mu_{A\otimes V})$ in $\mathsf{EM}^w(\mathcal{K})$ over the corresponding monad induced by $(A \otimes W, \mu_{A\otimes W})$, is a morphism*

$$\lambda : W \otimes V \to A \otimes V \otimes W$$

in $\mathcal{C}$ satisfying (52), (53), (54), (55) and

$$(\mu_A \otimes V \otimes W) \circ (\mu_A \otimes \psi_V^A \otimes W) \circ (A \otimes A \otimes V \otimes \sigma_W^A) \circ (A \otimes (\gamma_A \circ \nu_V) \otimes W) \circ \psi_W^A \circ (W \otimes \eta_A) \tag{58}$$

$$= (\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\psi_W^A \otimes V) \circ (W \otimes \nu_V)$$

$$(\mu_A \otimes V \otimes W) \circ (\mu_A \otimes \psi_V^A \otimes W) \circ (A \otimes \sigma_V^A \otimes \psi_W^A) \circ (\psi_V^A \otimes V \otimes W \otimes \eta_A) \circ (V \otimes (\gamma_A \circ \nu_V)) = \gamma_A \circ \psi_V^A \circ (V \otimes \eta_A), \tag{59}$$

where $\gamma_A = (\mu_A \otimes V \otimes W) \circ (A \otimes \lambda) \circ (\beta_{\nu_W} \otimes V)$.

Let $((F,\psi_F^S),\sigma_F^S,\nu_F)$, $((G,\psi_G^S),\sigma_G^S,\nu_G)$ be monads in $\mathsf{EM}^w(\mathcal{K})$. If λ is a distributive law of the monad $((F,\psi_F^S),\sigma_F^S,\nu_F)$ over the monad $((G,\psi_G^S),\sigma_G^S,\nu_G)$, we can obtain a new monad (the composite monad associated to λ (see [3]))

$$((F\otimes G,\psi_{F\otimes G}^S),\sigma_{F\otimes G}^S,\nu_{F\otimes G})$$

where

$$\psi_{F\otimes G}^S=(\psi_F^S\otimes G)\circ(F\otimes\psi_G^S), \tag{60}$$

$$\sigma_{F\otimes G}^S=(\sigma_G^S\circledast\sigma_F^S)\bullet(id_{(G,\psi_G^S)}\circledast\lambda\circledast id_{(F,\psi_F^S)}), \tag{61}$$

$$\nu_{F\otimes G}=\nu_G\circledast\nu_F. \tag{62}$$

Therefore, it is easy to show that (61) and (62) can be written as

$$\sigma_{F\otimes G}^S=(\mu_S\otimes F\otimes G)\circ(\mu_S\otimes\psi_F^S\otimes G)\circ(S\otimes\sigma_F^S\otimes\sigma_G^S)\circ(\psi_F^S\otimes F\otimes G\otimes G)\circ(F\otimes\lambda\otimes G), \tag{63}$$

$$\nu_{F\otimes G}=(\mu_S\otimes F\otimes G)\circ(S\otimes\psi_F^S\otimes G)\circ(\nu_F\otimes\nu_G). \tag{64}$$

As a consequence, if $(A\otimes V,\mu_{A\otimes V})$ and $(A\otimes W,\mu_{A\otimes W})$ are weak crossed products, with preunits ν_V and ν_W, and $\lambda: W\otimes V\to A\otimes V\otimes W$ is a distributive law of $(A\otimes V,\mu_{A\otimes V})$ over $(A\otimes W,\mu_{A\otimes W})$, we obtain a new weak crossed product $(A\otimes V\otimes W,\mu_{A\otimes V\otimes W})$ associated to the quadruple

$$\mathbb{A}_{V\otimes W}^{\lambda}=(A,V\otimes W,\psi_{V\otimes W}^A,\sigma_{V\otimes W}^A)$$

defined by

$$\psi_{V\otimes W}^A=(\psi_V^A\otimes W)\circ(V\otimes\psi_W^A),$$

$$\sigma_{V\otimes W}^A=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi_V^A\otimes W)\circ(A\otimes\sigma_V^A\otimes\sigma_W^A)\circ(\psi_V^A\otimes V\otimes W\otimes W)\circ(V\otimes\lambda\otimes W),$$

and with preunit

$$\nu_{V\otimes W}=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\nu_V\otimes\nu_W).$$

Following [27], for weak distributive laws between monads in $\mathsf{EM}^w(\mathcal{K})$, we have a similar construction but in this case we do not have a monad because the unit conditions are not always fulfilled.

From now on, if λ is a (weak) distributive law, the product associated to $\mathbb{A}_{V\otimes W}^{\lambda}$ will be called the λ-iterated product of $(A\otimes V,\mu_{A\otimes V})$ and $(A\otimes W,\mu_{A\otimes W})$.

2. A different way to iterate weak crossed products

The aim of this section is to iterate weak crossed products with a common monoid, that is, weak crossed products induced by quadruples of the form $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$ where A is fixed, from a different perspective to the one presented in the previous section.

Definition 2.1. Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$ and $\mathbb{A}_W=(A,W,\psi_W^A,\sigma_W^A)$ be two quadruples. We say that

$$\Delta_{V\otimes W}: V\otimes W\to V\otimes W$$

is a link morphism between $\mathbb{A}_V$ and $\mathbb{A}_W$ if the following conditions hold:

$$\Gamma_{V\otimes W}^A=(A\otimes\Delta_{V\otimes W})\circ\Gamma_{V\otimes W}^A, \tag{65}$$

$$\Gamma_{V\otimes W}^A=\nabla_{A\otimes V\otimes W}\circ\psi_{V\otimes W}^A, \tag{66}$$

where

$$\Gamma_{V\otimes W}^A=\psi_{V\otimes W}^A\circ(\Delta_{V\otimes W}\otimes A).$$

and $\nabla_{A\otimes V\otimes W}: A\otimes V\otimes W\to A\otimes V\otimes W$ is the morphism defined by

$$\nabla_{A\otimes V\otimes W}=(\mu_A\otimes V\otimes W)\circ(A\otimes\Gamma_{V\otimes W}^A)\circ(A\otimes V\otimes W\otimes\eta_A).$$

Lemma 2.2. *Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$ and $\mathbb{A}_W=(A,W,\psi_W^A,\sigma_W^A)$ be two quadruples. If there exists a link morphism $\Delta_{V\otimes W}: V\otimes W\to V\otimes W$ between them, the morphism $\Gamma_{V\otimes W}^A$ introduced in the previous definition satisfies (1) and as a consequence $\nabla_{A\otimes V\otimes W}$ is an idempotent morphism and the following identity holds:*

$$\Gamma_{V\otimes W}^A=\nabla_{A\otimes V\otimes W}\circ\Gamma_{V\otimes W}^A. \tag{67}$$

Proof:
Using that ψ_V^A, ψ_W^A satisfy (1) and $\Delta_{V\otimes W}$ satisfies (65) we obtain

$$\begin{aligned}
&(\mu_A\otimes V\otimes W)\circ(A\otimes\Gamma^A_{V\otimes W})\circ(\Gamma^A_{V\otimes W}\otimes A)\\
&=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\psi_V^A\otimes\psi_W^A)\circ(V\otimes\psi_W^A\otimes A)\circ(\Delta_{V\otimes W}\otimes A)\\
&=(\psi_V^A\otimes W)\circ(V\otimes\psi_W^A)\circ(\Delta_{V\otimes W}\otimes\mu_A)\\
&=\Gamma^A_{V\otimes W}\circ(V\otimes W\otimes\mu_A)
\end{aligned}$$

and then (1) holds for $\Gamma^A_{V\otimes W}$. Finally, (67) follows directly from (1) for $\Gamma^A_{V\otimes W}$.

□

Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$ and $\mathbb{A}_W=(A,W,\psi_W^A,\sigma_W^A)$ be two quadruples with a link morphism $\Delta_{V\otimes W}$ between them. Let, as in the previous section, $\mathcal{K}$ the one-object 2-category corresponding to $\mathcal{C}$. Note that, by the previous lemma, the pair $(V\otimes W,\Gamma^A_{V\otimes W})$ is a 1-cell in $\mathsf{EM}^w(\mathcal{K})$. Also, the morphism

$$p=\Gamma^A_{V\otimes W}\circ(V\otimes W\otimes\eta_A):V\otimes W\to A\otimes V\otimes W$$

is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$ between $(V\otimes W,\psi^A_{V\otimes W})$ and $(V\otimes W,\Gamma^A_{V\otimes W})$ because, by (65) and (1) for $\mathbb{A}_V$ and $\mathbb{A}_W$ we have

$$\begin{aligned}
&(\mu_A\otimes V\otimes W)\circ(A\otimes\Gamma^A_{V\otimes W})\circ(p\otimes\eta_A)\\
&=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\psi_V^A\otimes\psi_W^A)\circ(V\otimes\psi_W^A\otimes A)\circ(\Delta_{V\otimes W}\otimes\eta_A\otimes\eta_A)\\
&=\Gamma^A_{V\otimes W}\circ(V\otimes W\otimes\mu_A),
\end{aligned}$$

and, on the other hand, by (66) and (1) for $\Gamma^A_{V\otimes W}$

$$(\mu_A\otimes V\otimes W)\circ(A\otimes p)\circ\psi^A_{V\otimes W}=\Gamma^A_{V\otimes W}=(\mu_A\otimes V\otimes W)\circ(A\otimes\Gamma^A_{V\otimes W})\circ(p\otimes A).$$

Similary $i=p$ is a 2-cell in $\mathsf{EM}^w(\mathcal{K})$ between $(V\otimes W,\Gamma^A_{V\otimes W})$ and $(V\otimes W,\psi^A_{V\otimes W})$, and $p\bullet i=id_{(V\otimes W,\Gamma^A_{V\otimes W})}$. Therefore,

$$\Omega=i\bullet p:(V\otimes W,\psi^A_{V\otimes W})\Rightarrow(V\otimes W,\psi^A_{V\otimes W}) \tag{68}$$

is an idempotent 2-cell in $\mathsf{EM}^w(\mathcal{K})$.

Definition 2.3. Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$ and $\mathbb{A}_W=(A,W,\psi_W^A,\sigma_W^A)$ be two quadruples. We say that

$$\tau_W^V:W\otimes V\to V\otimes W$$

is a twisting morphism between $\mathbb{A}_V$ and $\mathbb{A}_W$ if the following conditions hold:

(i) $(\psi_V^A\otimes W)\circ(V\otimes\psi_W^A)\circ(\tau_W^V\otimes A)=(A\otimes\tau_W^V)\circ(\psi_W^A\otimes V)\circ(W\otimes\psi_V^A)$.

(ii) $(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma_V^A\otimes W)\circ(\psi_V^A\otimes\tau_W^V)\circ(V\otimes\sigma_W^A\otimes V)\circ(\tau_W^V\otimes W\otimes V)=$
$(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes\sigma_W^A)\circ(A\otimes\tau_W^V\otimes W)\circ(\psi_W^A\otimes V\otimes W)\circ(W\otimes\sigma_V^A\otimes W)\circ(W\otimes V\otimes\tau_W^V)$.

Theorem 2.4. *Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$, $\mathbb{A}_W=(A,V,\psi_W^A,\sigma_W^A)$ be two quadruples satisfying (4) and (5) with a link morphism $\Delta_{V\otimes W}:V\otimes W\to V\otimes W$ and with a twisting morphism $\tau_W^V:W\otimes V\to V\otimes W$ between them. Then if we define $\sigma^A_{V\otimes W}:V\otimes W\otimes V\otimes W\to A\otimes V\otimes W$ by*

$$\sigma^A_{V\otimes W}=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\sigma_V^A\otimes\sigma_W^A)\circ(V\otimes\tau_W^V\otimes W) \tag{69}$$

and it satisfies

$$\sigma^A_{V\otimes W}=\sigma^A_{V\otimes W}\circ(\Delta_{V\otimes W}\otimes V\otimes W), \tag{70}$$

$$\sigma^A_{V\otimes W}=\sigma^A_{V\otimes W}\circ(V\otimes W\otimes\Delta_{V\otimes W}), \tag{71}$$

$$\sigma^A_{V\otimes W}=(A\otimes\Delta_{V\otimes W})\circ\sigma^A_{V\otimes W}, \tag{72}$$

the quadruple $\mathbb{A}_{V\otimes W}=(A,V\otimes W,\Gamma^A_{V\otimes W},\sigma^A_{V\otimes W})$ satisfies the equalities (4), (5) and (10). As a consequence, $(A\otimes V\otimes W,\mu_{A\otimes V\otimes W})$ is a weak crossed product with

$$\mu_{A\otimes V\otimes W}=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\sigma^A_{V\otimes W})\circ(A\otimes\Gamma^A_{V\otimes W}\otimes V).$$

Proof:
First we prove the twisted condition.

$$
\begin{aligned}
&(\mu_A\otimes V)\circ(A\otimes\Gamma^A_{V\otimes W})\circ(\sigma^A_{V\otimes W}\otimes A)\\
&=(((\mu_A\otimes V)\circ(A\otimes\psi^A_V))\otimes W)\circ(\sigma^A_V\otimes((\mu_A\otimes W)\circ(A\otimes\psi^A_W)\circ(\sigma^A_W\otimes A)))\circ(V\otimes\tau^V_W\otimes W\otimes A)\\
&\quad\circ(\Delta_{V\otimes W}\otimes\Delta_{V\otimes W}\otimes A)\\
&=((\mu_A\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\circ(V\otimes\psi^A_V))\otimes W)\circ(V\otimes V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma^A_W)\circ(\psi^A_W\otimes W)\\
&\quad\circ(W\otimes\psi^A_W)))\circ(V\otimes\tau^V_W\otimes W\otimes A)\circ(\Delta_{V\otimes W}\otimes\Delta_{V\otimes W}\otimes A)\\
&=(((\mu_A\otimes V)\circ(\mu_A\otimes\sigma^A_V)\otimes(A\otimes\psi^A_V\otimes V)\circ(\psi^A_V\otimes\psi^A_V))\otimes W)\\
&\quad\circ(V\otimes((\psi^A_V\otimes\sigma^A_W)\circ(V\otimes\psi^A_W\otimes W)\circ(\tau^V_W\otimes\psi^A_W)))\circ(\Delta_{V\otimes W}\otimes\Delta_{V\otimes W}\otimes A)\\
&=(((\mu_A\otimes V)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\circ(V\otimes\psi^A_V))))\otimes W)\\
&\quad\circ(A\otimes V\otimes V\otimes\sigma^A_W)\circ(((\psi^A_V\otimes\tau^V_W)\circ(V\otimes\psi^A_W\otimes V)\circ(V\otimes W\otimes\psi^A_V))\otimes W)\\
&\quad\circ(V\otimes W\otimes V\otimes\psi^A_W)\circ(\Delta_{V\otimes W}\otimes\Delta_{V\otimes W}\otimes A)\\
&=(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_{V\otimes W})\circ(\Gamma^A_{V\otimes W}\otimes V\otimes W)\circ(V\otimes W\otimes\Gamma^A_{V\otimes W}).
\end{aligned}
$$

In the previous calculus, the first equality follows by, (70), (71), (72), (1) for $\mathbb{A}_V$ and the associativity of μ_A, the second one follows by the twisted condition for $\mathbb{A}_V$ and $\mathbb{A}_W$, the third one follows by (1) for $\mathbb{A}_V$ and the fourth one follows by (i) of Definition (2.3) as well as the associativity of μ_A. Finally, in the last one we use the twisted condition for $\mathbb{A}_V$.

The proof for the cocycle condition is the following:

$$
\begin{aligned}
&(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_{V\otimes W})\circ(\sigma^A_{V\otimes W}\otimes V\otimes W)\\
&=((\mu_A\circ(A\otimes\mu_A))\otimes V\otimes W)\circ(A\otimes\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes((A\otimes\sigma^A_V)\circ(\sigma^A_V\otimes V))\otimes A\otimes W)\\
&\quad\circ(\psi^A_V\otimes V\otimes V\otimes\sigma^A_W)\circ(V\otimes((\psi^A_V\otimes\tau^V_W)\circ(V\otimes\sigma^A_W\otimes V)\circ(\tau^V_W\otimes W\otimes V))\otimes W)\\
&\quad\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes\sigma^A_V\otimes A\otimes W)\circ(\psi^A_V\otimes V\otimes\sigma^A_W)\\
&\quad\circ(V\otimes[(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\circ(\psi^A_V\otimes\tau^V_W)\circ(V\otimes\sigma^A_W\otimes V)\circ(\tau^V_W\otimes W\otimes V)]\otimes W)\circ\\
&\quad\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes\sigma^A_V\otimes A\otimes W)\circ(\psi^A_V\otimes V\otimes\sigma^A_W)\\
&\quad\circ(V\otimes[(\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes\sigma^A_W)\circ(A\otimes\tau^V_W\otimes W)\circ(\psi^A_W\otimes V\otimes W)\\
&\quad\circ(V\otimes\sigma^A_V\otimes W)\circ(W\otimes V\otimes\tau^V_W)]\otimes W)\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\circ(V\otimes\psi^A_V))\otimes\sigma^A_W)\\
&\quad\circ(A\otimes V\otimes V\otimes\sigma^A_W\otimes W)\circ(\psi^A_V\otimes\tau^V_W\otimes W\otimes W)\circ(V\otimes\psi^A_W\otimes V\otimes W\otimes W)\\
&\quad\circ(V\otimes W\otimes((\sigma^A_V\otimes W)\circ(V\otimes\tau^V_W))\otimes W)\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi^A_V)\circ(\sigma^A_V\otimes V))\otimes\sigma^A_W)\\
&\quad\circ(A\otimes V\otimes V\otimes\sigma^A_W\otimes W)\circ(\psi^A_V\otimes\tau^V_W\otimes W\otimes V)\circ(V\otimes\psi^A_W\otimes V\otimes W\otimes W)\\
&\quad\circ(V\otimes W\otimes((\sigma^A_V\otimes W)\circ(V\otimes\tau^V_W))\otimes W)\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes\sigma^A_V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma^A_W)\circ(\sigma^A_W\otimes W)))\\
&\quad\circ(((\psi^A_V\otimes\tau^V_W)\circ(V\otimes\psi^A_W\otimes V)\circ(V\otimes W\otimes\sigma^A_V))\otimes W\otimes W)\circ(V\otimes W\otimes V\otimes\tau^V_W\otimes W)\\
&\quad\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes\sigma^A_V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma^A_W)\circ(\psi^A_W\otimes W)\circ(W\otimes\sigma^A_W)))\\
&\quad\circ(((\psi^A_V\otimes\tau^V_W)\circ(V\otimes\psi^A_W\otimes V)\circ(V\otimes W\otimes\sigma^A_V))\otimes W\otimes W)\circ(V\otimes W\otimes V\otimes\tau^V_W\otimes W)\\
&\quad\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi^A_V\otimes W)\\
&\quad\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi^A_V)\circ(\sigma^A_V\otimes A))\otimes\sigma^A_W)\circ(A\otimes V\otimes V\otimes\psi^A_W\otimes W)\\
&\quad\circ(\psi^A_V\otimes\tau^V_W\otimes\sigma^A_W)\circ(V\otimes\psi^A_W\otimes V\otimes W\otimes W)\circ(V\otimes W\otimes((\sigma^A_V\otimes W)\circ(V\otimes\tau^V_W))\otimes W)\\
&\quad\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi^A_V\otimes W)\\
&\quad\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\circ(V\otimes\psi^A_V))\otimes\sigma^A_W)\circ(A\otimes V\otimes V\otimes\psi^A_W\otimes W)\\
&\quad\circ(\psi^A_V\otimes\tau^V_W\otimes\sigma^A_W)\circ(V\otimes\psi^A_W\otimes V\otimes W\otimes W)\circ(V\otimes W\otimes((\sigma^A_V\otimes W)\circ(V\otimes\tau^V_W))\otimes W)\\
&\quad\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)\\
&=(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi^A_V\otimes W)\circ(\mu_A\circ\sigma^A_V\otimes\sigma^A_W)\\
&\quad\circ(A\otimes\psi^A_V\otimes\tau^V_W\otimes W)\circ(\psi^A_V\otimes\psi^A_W\otimes V\otimes W)\circ(V\otimes\psi^A_W\otimes\psi^A_V\otimes W)\circ(V\otimes W\otimes\sigma^A_V\otimes\sigma^A_W)
\end{aligned}
$$

$$\circ(V\otimes W\otimes V\otimes \tau_W^V\otimes W)\circ(\Delta_{V\otimes W}\otimes V\otimes W\otimes V\otimes W)$$
$$=(\mu_A\otimes V\otimes W)\circ(A\otimes \sigma_{V\otimes W}^A)\circ(\Gamma_{V\otimes W}^A\otimes V)\circ(V\otimes W\otimes \sigma_{V\otimes W}^A).$$

In this proof, the first equality follows by (70), the associativity of μ_A and the twisted condition for $\mathbb{A}_V$, the second one follows by the cocycle condition for $\mathbb{A}_V$, (1) for $\mathbb{A}_V$ and the associativity of μ_A. In the third one we used (ii) of Definition (2.3). The fourth one is a consequence of (1) for $\mathbb{A}_V$ and the associativity of μ_A. The fifth one follows by the twisted condition for $\mathbb{A}_V$ and the associativity of μ_A. In the sixth one we used (1) for $\mathbb{A}_V$ and the associativity of μ_A. The seventh one follows by the cocycle condition for $\mathbb{A}_W$ and in the eight one we applied (1) for $\mathbb{A}_V$ and the associativity of μ_A again. The ninth one follows by the twisted condition for $\mathbb{A}_V$ and the tenth one follows by (i) of Definition (2.3) and the associativity of μ_A. Finally, the last one was obtained using (1) for $\mathbb{A}_V$ and $\mathbb{A}_W$.

The proof for the equality (10) is the following:

$$\nabla_{A\otimes V\otimes W}\circ\sigma_{V\otimes W}^A$$
$$=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes\psi_W^A)\circ(((A\otimes\Delta_{V\otimes W})\circ\sigma_{V\otimes W}^A)\otimes\eta_A)$$
$$=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes\psi_W^A)\circ(\sigma_{V\otimes W}^A\otimes\eta_A)$$
$$=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\sigma_V^A\otimes(\nabla_{A\otimes W}\circ\sigma_W^A))\circ(V\otimes\tau_W^V\otimes W)$$
$$=\sigma_{V\otimes W}^A,$$

where the first equality follows by definition, the second one by (71), the third one by (1) for ψ_V^A and by the associativity of μ_A. The last one relies on the properties of σ_W^A, that is $\nabla_{A\otimes W}\circ\sigma_W^A=\sigma_W^A$.

□

Definition 2.5. Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$, $\mathbb{A}_W=(A,W,\psi_W^A,\sigma_W^A)$ be two quadruples satisfying (4) and (5) with a link morphism $\Delta_{V\otimes W}:V\otimes W\to V\otimes W$ and with a twisting morphism $\tau_W^V:W\otimes V\to V\otimes W$ between them. Let $(A\otimes V,\mu_{A\otimes V})$ and $(A\otimes W,\mu_{A\otimes W})$ be the weak crossed products associated to $\mathbb{A}_V$ and $\mathbb{A}_W$ and suppose that the morphism $\sigma_{V\otimes W}^A$ defined in (69) satisfies (70), (71) and (72). The weak crossed product $(A\otimes V\otimes W,\mu_{A\otimes V\otimes W})$ defined in the previous theorem will be called the iterated weak crossed product of $(A\otimes V,\mu_{A\otimes V})$ and $(A\otimes W,\mu_{A\otimes W})$.

In the following theorem we introduce the conditions that implies the existence of a preunit for the iterated weak crossed product defined previously.

Theorem 2.6. *Let $\mathbb{A}_V=(A,V,\psi_V^A,\sigma_V^A)$, $\mathbb{A}_W=(A,W,\psi_W^A,\sigma_W^A)$ be two quadruples satisfying (4) and (5) with a link morphism $\Delta_{V\otimes W}:V\otimes W\to V\otimes W$ and with a twisting morphism $\tau_W^V:W\otimes V\to V\otimes W$ between them. Let $(A\otimes V,\mu_{A\otimes V})$ and $(A\otimes W,\mu_{A\otimes W})$ be the weak crossed products associated to $\mathbb{A}_V$ and $\mathbb{A}_W$ and suppose that $\nu_V:K\to A\otimes V$ and $\nu_W:K\to A\otimes W$ are preunits for $\mu_{A\otimes V}$ and $\mu_{A\otimes W}$. If the morphism $\sigma_{V\otimes W}^A$ defined in (69) satisfies (70), (71), (72) and the following equalities hold*

$$(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma_V^A\otimes W)\circ(\psi_V^A\otimes\tau_W^V)\circ(V\otimes\psi_W^A\otimes V)\circ(V\otimes W\otimes\nu_V)\tag{73}$$
$$=\nabla_{A\otimes V\otimes W}\circ(\eta_A\otimes V\otimes W),$$
$$(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes\sigma_W^A)\circ(A\otimes\tau_W^V\otimes W)\circ(\nu_W\otimes V\otimes W)\tag{74}$$
$$=\nabla_{A\otimes V\otimes W}\circ(\eta_A\otimes V\otimes W),$$

the iterated weak crossed product of $(A\otimes V,\mu_{A\otimes V})$ and $(A\otimes W,\mu_{A\otimes W})$ has a preunit defined by

$$\nu_{V\otimes W}=\nabla_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\nu_V\otimes\nu_W).\tag{75}$$

Proof:
Note that to prove that $\nu_{V\otimes W}$ is a preunit we need to show that the equalities (17), (18) and (19) hold for the quadruple $\mathbb{A}_{V\otimes W}=(A,V\otimes W,\Gamma_{V\otimes W}^A,\sigma_{V\otimes W}^A)$.

In this setting, the equality (19) holds because:

$$(\mu_A\otimes V\otimes W)\circ(A\otimes\Gamma_{V\otimes W}^A)\circ(\nu_{V\otimes W}\otimes A)$$
$$=\nabla_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes W)\circ(A\otimes\psi_V^A\otimes\psi_W^A)$$
$$\circ(\nu_V\otimes\nu_W\otimes A)$$

$$
\begin{aligned}
&= \nabla_{A\otimes V\otimes W}\circ(\beta_{\nu_V}\otimes W)\circ\beta_{\nu_W}\\
&= \nabla_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes((\beta_{\nu_V}\otimes W)\circ\nu_W))\\
&= \beta_{\nu_{V\otimes W}},
\end{aligned}
$$

where the first equality follows by (3) for $\mathbb{A}_{V\otimes W}$, (66), and the the left A-linearity of $\nabla_{A\otimes V\otimes W}$, the second one follows by the associativity of μ_A, (1) for $\mathbb{A}_V$ and (19) for β_{ν_V} and β_{ν_W}, the third one follows by the left A-linearity of β_{ν_V} and the last one follows from the left A-linearity of $\nabla_{A\otimes V\otimes W}$.

The proof for the equality (17) is the following:

$$
\begin{aligned}
&(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_{V\otimes W})\circ(\Gamma^A_{V\otimes W}\otimes V\otimes W)\circ(V\otimes W\otimes\nu_{V\otimes W})\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_{V\otimes W})\circ(\Gamma^A_{V\otimes W}\otimes V\otimes W)\circ(V\otimes W\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(\nu_V\otimes\nu_W)))\\
&= (\mu_A\otimes V\otimes W)\circ(\mu_A\otimes A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes\sigma^A_V\otimes W)\circ(A\otimes A\otimes\psi^A_V\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes\psi^A_V\otimes W)\\
&\quad\circ(A\otimes V\otimes A\otimes V\otimes\sigma^A_W)\circ(A\otimes V\otimes A\otimes\tau^V_W\otimes W)\circ(\psi^A_V\otimes\psi^A_W\otimes V\otimes W)\circ(V\otimes\psi^A_W\otimes\psi^A_V\otimes W)\\
&\quad\circ(V\otimes W\otimes\nu_V\otimes\nu_W)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\circ((\psi^A_V\circ(V\otimes(\mu_A\circ(A\otimes\mu_A))))\otimes V\otimes W)\circ(V\otimes A\otimes A\otimes\psi^A_V\otimes W)\\
&\quad\circ(V\otimes A\otimes A\otimes V\otimes\sigma^A_W)\circ(V\otimes A\otimes((A\otimes\tau^V_W)\circ(\psi^A_W\otimes V)\circ(W\otimes\psi^A_V))\otimes W)\circ(V\otimes\psi^A_W\otimes V\otimes A\otimes W)\\
&\quad\circ(V\otimes W\otimes\nu_V\otimes\nu_W)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\circ((\psi^A_V\circ(V\otimes(\mu_A\circ(A\otimes\mu_A))))\otimes V\otimes W)\circ(V\otimes A\otimes A\otimes\psi^A_V\otimes W)\\
&\quad\circ(V\otimes A\otimes A\otimes V\otimes\sigma^A_W)\circ(V\otimes A\otimes(((\psi^A_V\otimes W)\circ(V\otimes\psi^A_W)\circ(\tau^V_W\otimes A))\otimes W)\circ(V\otimes\psi^A_W\otimes V\otimes A\otimes W)\\
&\quad\circ(V\otimes W\otimes\nu_V\otimes\nu_W)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\circ((\psi^A_V\circ(V\otimes\mu_A))\otimes V\otimes W)\circ(V\otimes A\otimes\psi^A_V\otimes W)\\
&\quad\circ(V\otimes A\otimes V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma^A_W)\circ(\psi^A_W\otimes W)\circ(W\otimes\nu_W)))\circ(V\otimes A\otimes\tau^V_W)\circ(V\otimes\psi^A_W\otimes V)\\
&\quad\circ(V\otimes W\otimes\nu_V)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\circ(\psi^A_V\otimes V\otimes W)\circ(V\otimes\mu_A\otimes V\otimes W)\circ(V\otimes A\otimes\psi^A_V\otimes W)\circ(V\otimes A\otimes V\otimes\psi^A_W)\\
&\quad\circ(V\otimes A\otimes\tau^V_W\otimes A)\circ(V\otimes\psi^A_W\otimes V\otimes A)\circ(V\otimes W\otimes\nu_V\otimes\eta_A)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\circ(V\otimes\psi^A_V))\otimes W)\circ(A\otimes V\otimes V\otimes\psi^A_W)\circ(\psi^A_V\otimes\tau^V_W\otimes A)\\
&\quad\circ(V\otimes\psi^A_W\otimes V\otimes A)\circ(V\otimes W\otimes\nu_V\otimes\eta_A)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes(\psi^A_W\circ(W\otimes\eta_A)))\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\\
&\quad\circ(\psi^A_V\otimes\tau^V_W)\circ(V\otimes\psi^A_W\otimes V)\circ(V\otimes W\otimes\nu_V)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes(\psi^A_W\circ(W\otimes\eta_A)))\circ(\psi^A_V\otimes W)\circ(V\otimes\psi^A_W)\circ(\Delta_{V\otimes W}\otimes\eta_A)\\
&= \nabla_{A\otimes V\otimes W}\circ(\eta_A\otimes V\otimes W).
\end{aligned}
$$

In this proof, the first equality follows by (8) for $\mathbb{A}_{V\otimes W}$, the second one follows by (1) for $\mathbb{A}_V$, $\mathbb{A}_W$, the associativity of μ_A, the twisted condition for $\mathbb{A}_V$, (66) and by (9) for $\mathbb{A}_{V\otimes W}$, the third one follows by (1) for $\mathbb{A}_V$ and the associativity of μ_A. In the fourth one we applied (i) of the definition of twisting morphism and the fifth one is a consequence of (1) for $\mathbb{A}_V$. The sixth one follows by (17) for $\mathbb{A}_W$, the seventh one follows by (1) for $\mathbb{A}_V$ and the ssociativity of μ_A, the eight one relies on the associativity of μ_A and the twisted condition for $\mathbb{A}_V$. Finally, the ninth one is a consequence of (73) and the last one follows by (1) for $\mathbb{A}_V$, $\mathbb{A}_W$.

On the other hand, the proof for the identity (18) is

$$
\begin{aligned}
&(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_{V\otimes W})\circ(\nu_{V\otimes W}\otimes V\otimes W)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_{V\otimes W})\circ(((\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(\nu_V\otimes\nu_W))\otimes V\otimes W)\\
&= (\mu_A\otimes V\otimes W)\circ((\mu_A\circ(A\otimes\mu_A))\otimes\sigma^A_V\otimes W)\circ(A\otimes A\otimes\psi^A_V\otimes V\otimes W)\circ(A\otimes A\otimes V\otimes\psi^A_V\otimes W)\\
&\quad\circ(A\otimes A\otimes V\otimes V\otimes\sigma^A_W)\circ(A\otimes\psi^A_V\otimes\tau^V_W\otimes W)\circ(\nu_V\otimes\nu_W\otimes V\otimes W)\\
&= (\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\sigma^A_V\otimes W)\circ(A\otimes\psi^A_V\otimes V\otimes W)\circ(\nu_V\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes\sigma^A_W)\\
&\quad\circ(A\otimes\tau^V_W\otimes W)\circ(\nu_W\otimes V\otimes W)\\
&= (\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\sigma^A_V\otimes W)\circ(A\otimes\psi^A_V\otimes V\otimes W)\circ(\nu_V\otimes(\nabla_{A\otimes V\otimes W}\circ(\eta_A\otimes V\otimes W)))\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes(\psi^A_W\circ(W\otimes\eta_A)))\circ(((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\nu_V\otimes V))\otimes W)\\
&\quad\circ\Delta_{V\otimes W}\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes(\psi^A_W\circ(W\otimes\eta_A)))\circ((\nabla_{A\otimes V}\circ(\eta_A\otimes V))\otimes W)\circ\Delta_{V\otimes W}\\
&= \nabla_{A\otimes V\otimes W}\circ(\eta_A\otimes V\otimes W).
\end{aligned}
$$

The first equality follows by (9) for $\mathbb{A}_{V\otimes W}$, the second one follows by the associativity of μ_A and the twisted condition for $\mathbb{A}_V$, the third one follows by (1) for $\mathbb{A}_V$, the fourth one relies on (74), the fifth one follows by the twisted condition for $\mathbb{A}_V$ and the associativity of μ_A, the sixth one is a consequence of (18) for $\mathbb{A}_V$ and finally, the last one follows by (1) for $\mathbb{A}_V$.

□

Remark 2.7. Note that we can obtain similar results about the iteration process if we work with quadruples ${}_V\mathbb{A} = (V, A, \psi_A^V, \sigma_A^V)$ where $\psi_A^V : A \otimes V \to V \otimes A$ and $\sigma_A^V : V \otimes V \to V \otimes A$ satisfy the suitable conditions that define a weak crossed product on $V \otimes A$.

Theorem 2.8. *Let $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$, $\mathbb{A}_W = (A, W, \psi_W^A, \sigma_W^A)$ be two quadruples satisfying (4) and (5) with a link morphism $\Delta_{V\otimes W} : V \otimes W \to V \otimes W$ and with a twisting morphism $\tau_W^V : W \otimes V \to V \otimes W$ between them. Let $(A\otimes V, \mu_{A\otimes V})$ and $(A\otimes W, \mu_{A\otimes W})$ be the weak crossed products associated to $\mathbb{A}_V$ and $\mathbb{A}_W$ and suppose that $\nu_V : K \to A \otimes V$ and $\nu_W : K \to A \otimes W$ are preunits for $\mu_{A\otimes V}$ and $\mu_{A\otimes W}$. If the morphism $\sigma_{V\otimes W}^A$ defined in (69) satisfies (70), (71), (72) and the equalities (73), (74) hold, the triple $((V \otimes W, \psi_{V\otimes W}^A), \sigma_{V\otimes W}^A, \nu_{V\otimes W})$ is a premonad in $\mathsf{EM}^w(\mathcal{K})$, where $\nu_{V\otimes W}$ is the morphism introduced in (75). Also, the monad $((V \otimes W, \Gamma_{V\otimes W}^A), \sigma_{V\otimes W}^A, \nu_{V\otimes W})$ is the canonical retract monad induced by the idempotent 2-cell Ω defined in (68).*

Proof. The equality (28) follows because $(V\otimes W, \psi_{V\otimes W}^A)$ is a 1-cell in $\mathsf{EM}^w(\mathcal{K})$. The proofs for (29), (30) and (31) are similar with the ones used in Theorem 2.4, for $(V \otimes W, \Gamma_{V\otimes W}^A)$ and $\sigma_{V\otimes W}^A$, removing the linking morphism. Also, by (1) for $\mathbb{A}_V$ and $\mathbb{A}_W$, (3) for $\Gamma_{V\otimes W}^A$, and using that $\nu_{V\otimes W}$ is a preunit for the iterated weak crossed product $(A \otimes V \otimes W, \mu_{A\otimes V\otimes W})$ (see the proof of (19) in Theorem 2.6) we have

$$\begin{aligned}
&(\mu_A \otimes V \otimes W) \circ (A \otimes \psi_{V\otimes W}^A) \circ (\nu_{V\otimes W} \otimes A)\\
&= (\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes\psi_W^A)\circ(((\mu_A\otimes\Delta_{V\otimes W})\circ(A\otimes\psi_V^A\otimes W)\circ(\nu_V\otimes\nu_W))\otimes A)\\
&= (\mu_A \otimes V \otimes W) \circ (A \otimes \Gamma_{V\otimes W}^A) \circ (\nu_{V\otimes W} \otimes A)\\
&= (\mu_A \otimes V \otimes W) \circ (A \otimes (\nu_{V\otimes W})
\end{aligned}$$

and then (34) holds.

On the other hand, by (18) for the iterated weak crossed product $(A \otimes V \otimes W, \mu_{A\otimes V\otimes W})$ we have

$$(\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ (\nu_{V\otimes W} \otimes V \otimes W) = \nabla_{A\otimes V\otimes W} \circ (\eta_A \otimes V \otimes W)$$

Then, (35) holds, because by (9) for $\sigma_{V\otimes W}^A$, (66), and (17) for the iterated weak crossed product $(A \otimes V \otimes W, \mu_{A\otimes V\otimes W})$, we obtain

$$\begin{aligned}
&(\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ (\psi_{V\otimes W}^A \otimes V \otimes W) \circ (V \otimes W \otimes \nu_{V\otimes W})\\
&= (\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ ((\nabla_{A\otimes V\otimes W} \circ \psi_{V\otimes W}^A) \otimes V \otimes W) \circ (V \otimes W \otimes \nu_{V\otimes W})\\
&= (\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ (\Gamma_{V\otimes W}^A \otimes V \otimes W) \circ (V \otimes W \otimes \nu_{V\otimes W})\\
&= \nabla_{A\otimes V\otimes W} \circ (\eta_A \otimes V \otimes W).
\end{aligned}$$

Moreover, using (18) for the iterated weak crossed product $(A\otimes V\otimes W, \mu_{A\otimes V\otimes W})$, the left A-linearity of $\nabla_{A\otimes V\otimes W}$, and (22) for $\nabla_{A\otimes V\otimes W}$ and $\nu_{V\otimes W}$, we prove (36). Similarly, by (18) for the iterated weak crossed product $(A \otimes V \otimes W, \mu_{A\otimes V\otimes W})$, and (9) for $\sigma_{V\otimes W}^A$,

$$\begin{aligned}
&(\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ (((\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ (\nu_{V\otimes W} \otimes V \otimes W)) \otimes \otimes V \otimes W)\\
&= (\mu_A \otimes V \otimes W) \circ (A \otimes \sigma_{V\otimes W}^A) \circ ((\nabla_{A\otimes V\otimes W} \circ \circ(\eta_A \otimes V \otimes W)) \otimes V \otimes W)\\
&= \sigma_{V\otimes W}^A
\end{aligned}$$

and then (37) holds.

Finally, by similar arguments and the twisted condition for the iterated weak crossed product $(A \otimes V \otimes W, \mu_{A\otimes V\otimes W})$, we can prove the identities

$$\sigma_{V\otimes W}^A = p \bullet \sigma_{V\otimes W}^A \bullet (i \circledast i),$$

and

$$p \bullet \nu_{V\otimes W} = \nu_{V\otimes W}.$$

Therefore, the monad $((V\otimes W,\Gamma^A_{V\otimes W}),\sigma^A_{V\otimes W},\nu_{V\otimes W})$ is the canonical retract monad induced by the idempotent 2-cell Ω. □

In the previous theorem we find premonad in $\mathsf{EM}^w(\mathcal{K})$ defined by $((V\otimes W,\psi^A_{V\otimes W}),\sigma^A_{V\otimes W},\nu_{V\otimes W})$. For this premonad the following equality holds

$$\sigma^A_{V\otimes W}\bullet(id_{(W\otimes V\otimes W,\psi^A_{W\otimes V\otimes W})}\circledast\sigma^A_V)=(id_{(W,\psi^A_W)}\circledast\sigma^A_V)\bullet(\sigma^A_{V\otimes W}\circledast id_{(V,\psi^A_V)}). \tag{76}$$

Indeed:

$$\begin{array}{l}(id_{(W,\psi^A_W)}\circledast\sigma^A_V)\bullet(\sigma^A_{V\otimes W}\circledast id_{(V,\psi^A_V)})\\
=(\mu_A\otimes V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi^A_V)\circ(\sigma^A_V\otimes A))\otimes W)\circ(\psi^A_V\otimes V\otimes\psi^A_W)\circ(V\otimes\sigma^A_{V\otimes W}\otimes\eta_A)\\
=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi^A_V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes A)\circ(A\otimes\psi^A_V))\otimes A\otimes W)\\
\circ(\psi^A_V\otimes V\otimes A\otimes\psi^A_W)\circ(V\otimes((\sigma^A_V\otimes\sigma^A_W)\circ(V\otimes\tau^V_W\otimes W))\otimes\eta_A)\\
=(\mu_A\otimes V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi^A_V)\circ(\psi^A_V\otimes A))\otimes W)\circ(((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\\
\circ(V\otimes\sigma^A_V))\otimes A\otimes\psi^A_W)\circ(V\otimes V\otimes((V\otimes\sigma^A_W)\circ(\tau^V_W\otimes W))\otimes\eta_A)\\
=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\sigma^A_V\otimes V))\otimes((\mu_A\otimes W)\circ(A\otimes\psi^A_W)\\
\circ(\sigma^A_W\otimes A)))\circ(V\otimes V\otimes\tau^V_W\otimes W\otimes\eta_A)\\
=(\mu_A\otimes V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi^A_V)\circ(\sigma^A_V\otimes A))\otimes W)\circ(A\otimes V\otimes V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma^A_W)\\
\circ(\psi^A_W\otimes A)\circ(A\otimes\psi^A_W)))\circ(\sigma^A_V\otimes\tau^V_W\otimes W\otimes\eta_A)\\
=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes((\mu_A\otimes W)\circ(A\otimes\sigma^A_W)\circ(\psi^A_W\otimes A)\circ(A\otimes\psi^A_W))\otimes W)\circ(A\otimes\psi^A_V\otimes V\otimes\sigma^A_W)\\
\circ(\sigma^A_V\otimes((\psi^A_V\otimes W)\circ(V\otimes\psi^A_W)\circ(\tau^V_W\otimes A))\otimes W)\circ(V\otimes V\otimes W\otimes V\otimes((\psi^A_W\otimes A)\circ(W\otimes\eta_A)))\\
=(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\sigma^A_{V\otimes W})\circ(A\otimes\psi^A_V\otimes W\otimes V\otimes W)\circ(\sigma^A_V\otimes((\psi^A_W\otimes V\otimes W)\circ(W\otimes\psi^A_V\otimes W)\\
\circ(W\otimes V\otimes((\psi^A_W\otimes A)\circ(W\otimes\eta_A))))\\
=\sigma^A_{V\otimes W}\bullet(id_{(W\otimes V\otimes W,\psi^A_{W\otimes V\otimes W})}\circledast\sigma^A_V)\end{array}$$

In the last equalities, the first one follows by (1) and (4) for $\mathbb{A}_V$. The second and the sixth ones follow by (1) for $\mathbb{A}_V$ and by the associativity of μ_A. The third one relies on (4) for $\mathbb{A}_V$ and on the associativity of μ_A. In the fourth one we used (5) and (1) for $\mathbb{A}_V$. The fifth one is a consequence of the the associativity of μ_A and (5) for $\mathbb{A}_W$. Finally, the seventh one follows by (4) for $\mathbb{A}_V$ and (i) of Definition 2.3, and the eighth one relies on (1) for $\mathbb{A}_V$ and $\mathbb{A}_W$, and on (4) for $\mathbb{A}_V$.

The previous equality is the equality (2.12) of [5]. Then, by Theorem 2.3 of [5], the premonad introduced in Theorem 2.8 corresponds to a monad in $\mathsf{EM}^w(\mathsf{EM}^w(\mathcal{K}))$ whose constitutent 1-cell is

$$\psi=\sigma^A_{V\otimes W}\bullet(id_{(W,\psi^A_W)}\circledast\sigma^A_V\circledast id_{(V\otimes W,\psi^A_{V\otimes W})})\bullet(\nu_{V\otimes W}\circledast id_{(W\otimes V\otimes W,\psi^A_{W\otimes V\otimes W})}) \tag{77}$$

$$\bullet(id_{(W\otimes V,\psi^A_{W\otimes V})}\circledast\nu_V).$$

Then, by (1) for $\mathbb{A}_V$ and $\mathbb{A}_W$, (18) and (3) for $\mathbb{A}_V$, and the associativity of μ_A, we have

$$\psi=(\mu_A\otimes V\otimes W)\circ(A\otimes((\psi^A_V\otimes W)\circ(V\otimes\sigma^A_W)\circ(\tau^V_W\otimes W)))\circ(\psi^A_W\otimes V\otimes W) \tag{78}$$

$$\circ(W\otimes((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V))\otimes W)\circ(W\otimes V\otimes\nu_{V\otimes W}):W\otimes V\to A\otimes V\otimes W.$$

Remark 2.9. Note that, if $\mathbb{A}_V=(A,V,\psi^A_V,\sigma^A_V)$, $\mathbb{A}_W=(A,W,\psi^A_W,\sigma^A_W)$ are two quadruples such that $\mathbb{A}_V$ satisfies the twisted condition (4) and there exists a morphism $\tau^V_W:W\otimes V\to V\otimes W$ satisfying

$$(\psi^A_V\otimes W)\circ(V\otimes\sigma^A_W)\circ(\tau^V_W\otimes W)\circ(W\otimes\tau^V_W)=(A\otimes\tau^V_W)\circ(\sigma^A_W\otimes V), \tag{79}$$

and

$$(\sigma^A_V\otimes W)\circ(V\otimes\tau^V_W)\circ(\tau^V_W\otimes V)=(A\otimes\tau^V_W)\circ(\psi^A_W\otimes V)\circ(W\otimes\sigma^A_V), \tag{80}$$

the equality (ii) of Definition 2.3 holds because:

$$\begin{array}{l}(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma^A_V\otimes W)\circ(\psi^A_V\otimes\tau^V_W)\circ(V\otimes\sigma^A_W\otimes V)\circ(\tau^V_W\otimes W\otimes V)\\
=(((\mu_A\otimes V)\circ(A\otimes\sigma^A_V)\circ(\psi^A_V\otimes V)\circ(V\otimes\psi^A_V))\otimes W)\circ(V\otimes V\otimes\sigma^A_W)\circ(V\otimes\tau^V_W\otimes W)\circ(\tau^V_W\otimes\tau^V_W)\\
=(((\mu_A\otimes V)\circ(A\otimes\psi^A_V)\circ(\sigma^A_V\otimes A))\otimes W)\circ(V\otimes V\otimes\sigma^A_W)\circ(V\otimes\tau^V_W\otimes W)\circ(\tau^V_W\otimes\tau^V_W)\\
=(\mu_A\otimes V\otimes W)\circ(A\otimes\psi^A_V\otimes W)\circ(A\otimes V\otimes\sigma^A_W)\circ(A\otimes\tau^V_W\otimes W)\circ(\psi^A_W\otimes V\otimes W)\circ(V\otimes\sigma^A_V\otimes W)\\
\circ(W\otimes V\otimes\tau^V_W)\end{array}$$

where the first equality follows by (79), the second one by (4) for the quadruple $\mathbb{A}_V$ and the last one by (80).

3. Some examples

The aim of this section is to provide some examples of the iteration process introduced in the previous ones.

Example 3.1. The category of endofunctors of $\mathcal{C}$ is a strict monoidal category with the composition of functors, denoted by $\otimes$, as the tensor product and the identity functor as the unit. We denote this category by $End(\mathcal{C})$. The morphisms in $End(\mathcal{C})$ are natural transformations between endofunctors and we denote the composition (the vertical composition) of these morphisms by $\circ$. The tensor product of morphisms in $End(\mathcal{C})$ is defined by the horizontal composition of natural transformations and in this paper is denoted by the same symbol used for the composition of functors (see [21] for the details of the horizontal and vertical compositions). Note that, if $\mathcal{C}$ is a category where every idempotent morphism splits it is easy to show that every idempotent morphism splits in $End(\mathcal{C})$. Given objects S, T, D and a morphism $\tau : T \to D$, we write $S \circledcirc \tau$ for $id_S \circledcirc \tau$ and $\tau \circledcirc S$ for $\tau \circledcirc id_S$ where id_S denotes the identity morphism for the object S.

A monad on $\mathcal{C}$ consists of a endofunctor $S : \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\eta_S : id_{\mathcal{C}} \to S$ (where $id_{\mathcal{C}}$ denotes the identity functor on $\mathcal{C}$) and $\mu_S : S^2 = S \circledcirc S \to S$. These are required to fulfill the following conditions

$$\mu_S \circ (S \circledcirc \eta_S) = \mu_S \circ (\eta_S \circledcirc S) = id_S, \tag{81}$$

$$\mu_S \circ (S \circledcirc \mu_S) = \mu_S \circ (\mu_S \circledcirc S). \tag{82}$$

Then, a monad on $\mathcal{C}$ can alternatively be defined as a monoid in the strict monoidal category $End(\mathcal{C})$.

The notion of wreath was introduced by Lack and Street in [19]. A monad S in $\mathcal{C}$ is a wreath if there exist an object in $T \in End(\mathcal{C})$ and morphisms in $End(\mathcal{C})$, $\psi : T \circledcirc S \to S \circledcirc T$, $\tau : id_{\mathcal{C}} \to S \circledcirc T$ and $v : T \circledcirc T \to S \circledcirc T$ satisfying the following conditions:

$$(\mu_S \circledcirc T) \circ (S \circledcirc \psi) \circ (\psi \circledcirc S) = \psi \circ (T \circledcirc \mu_S), \tag{83}$$

$$\psi \circ (T \circledcirc \eta_S) = \eta_S \circledcirc T, \tag{84}$$

$$(\mu_S \circledcirc T) \circ (S \circledcirc \tau) = (\mu_S \circledcirc T) \circ (S \circledcirc \psi) \circ (\tau \circledcirc S), \tag{85}$$

$$(\mu_S \circledcirc T) \circ (S \circledcirc v) \circ (\psi \circledcirc T) \circ (T \circledcirc \psi) = (\mu_S \circledcirc T) \circ (S \circledcirc \psi) \circ (v \circledcirc S), \tag{86}$$

$$(\mu_S \circledcirc T) \circ (S \circledcirc v) \circ (v \circledcirc T) = (\mu_S \circledcirc T) \circ (S \circledcirc v) \circ (\psi \circledcirc T) \circ (T \circledcirc v), \tag{87}$$

$$(\mu_S \circledcirc T) \circ (S \circledcirc v) \circ (\tau \circledcirc T) = \eta_S \circledcirc T = (\mu_S \circledcirc T) \circ (S \circledcirc v) \circ (\psi \circledcirc T) \circ (T \circledcirc \tau). \tag{88}$$

If we put $\psi_T^S = \psi$ and $\sigma_T^S = v$, we obtain that $\mathbb{S}_T = (S, T, \psi_T^S, \sigma_T^S)$ is a quadruple satisfying (1), (4) and (5) where the associated idempotent defined in (2) is $\nabla_{S \circledcirc T} = id_{S \circledcirc T}$ because ψ satisfies the identity (84). Then, the product induced by a wreath (wreath product) defined by

$$\mu_{S \circledcirc T} = (\mu_S \circledcirc T) \circ (\mu_S \circledcirc v) \circ (S \circledcirc \psi \circledcirc T)$$

is the one defined in (11) and it is associative because satisfies (iv) (twisted condition) and (v) (cocycle condition). Then $S \circledcirc T$ is a monad with unit $\eta_{S \circledcirc T} = \tau$.

An example of wreath products cames from the notion of distributive law introduced by Beck in [4] (see also [26]). Suppose that T and S are two monads on $\mathcal{C}$. A distributive law of the monad S over the monad T is a natural transformation

$$\lambda : T \circledcirc S \to S \circledcirc T$$

such that

$$\lambda \circ (\mu_T \circledcirc S) = (S \circledcirc \mu_T) \circ (\lambda \circledcirc T) \circ (T \circledcirc \lambda), \tag{89}$$

$$\lambda \circ (\eta_T \circledcirc S) = S \circledcirc \eta_T, \tag{90}$$

$$\lambda \circ (T \circledcirc \mu_S) = (\mu_S \circledcirc T) \circ (S \circledcirc \lambda) \circ (\lambda \circledcirc S), \tag{91}$$

$$\lambda \circ (T \circledcirc \eta_S) = \eta_S \circledcirc T. \tag{92}$$

Then, if $\tau = \eta_S \circledcirc \eta_T$ and $v = \eta_S \circledcirc \mu_T$ we obtain a wreath for the monad S and also a weak crossed product associated to the quadruple $\mathbb{S}_T = (S, T, \psi_T^S, \sigma_T^S)$ where $\psi_T^S = \lambda$, $\sigma_T^S = v$ and

$$\mu_{S\circledcirc T} = (\mu_S \circledcirc \mu_T) \circ (S \circledcirc \lambda \circledcirc T).$$

Suppose that S, T and D are monads in $\mathcal{C}$ such that there exists the following distributive laws between them

$$\lambda_1 : T \circledcirc S \to S \circledcirc T, \quad \lambda_2 : D \circledcirc T \to T \circledcirc D, \quad \lambda_3 : D \circledcirc S \to S \circledcirc D,$$

satisfying the compatibility identity (called the Yang-Baxter relation or the hexagon equation)

$$(S \circledcirc \lambda_2) \circ (\lambda_3 \circledcirc T) \circ (D \circledcirc \lambda_1) = (\lambda_1 \circledcirc D) \circ (T \circledcirc \lambda_3) \circ (\lambda_2 \circledcirc S). \tag{93}$$

Then, under these conditions we have two quadruples

$$\mathbb{S}_T = (S, T, \psi_T^S = \lambda_1, \sigma_T^S = \eta_S \circledcirc \mu_T),$$
$$\mathbb{S}_D = (S, D, \psi_D^S = \lambda_3, \sigma_D^S = \eta_S \circledcirc \mu_D),$$

satisfying (1), (4), (5). If we put $\Delta_{T\circledcirc D} = id_{T\circledcirc D}$ as a link morphism (note that in this case the equalities (70), (71) and (72) are trivial) and $\tau_D^T = \lambda_2$ we have that the condition (i) of Definition 2.3 holds because we assume (93). On the other hand, the condition (ii) of the same Definition also holds because:

$$\begin{aligned}&(\mu_S \circledcirc T \circledcirc D) \circ (S \circledcirc \sigma_T^S \circledcirc D) \circ (\psi_T^S \circledcirc \tau_D^T) \circledcirc (T \circledcirc \sigma_D^S \circledcirc T) \circ (\tau_D^T \circledcirc D \circledcirc T)\\ &= (\eta_S \circledcirc ((\mu_T \circledcirc \mu_D) \circ (T \circledcirc \lambda_2 \circledcirc D) \circ (\lambda_2 \circledcirc \lambda_2)))\\ &= (\mu_S \circledcirc T \circledcirc D) \circ (S \circledcirc \psi_T^S \circledcirc D) \circ (S \circledcirc T \circledcirc \sigma_D^S) \circ (S \circledcirc \tau_D^T \circledcirc D) \circ (\psi_D^S \circledcirc T \circledcirc D) \circ (D \circledcirc \sigma_T^S \circledcirc D) \circ\\ &\circ (D \circledcirc T \circledcirc \tau_D^T).\end{aligned}$$

Therefore, $\tau_D^T = \lambda_2$ is a twisting morphism between the quadruples $\mathbb{S}_T$ and $\mathbb{S}_D$. As a consequence, by Lemma 2.2 and Theorem 2.4, the quadruple

$$\mathbb{S}_{T\circledcirc D} = (S, T \circledcirc D, \psi_{T\circledcirc D}^S, \sigma_{T\circledcirc D}^S),$$

where

$$\psi_{T\circledcirc D}^S = (\psi_T^S \circledcirc D) \circ (T \circledcirc \psi_D^S) = (\lambda_1 \circledcirc D) \circ (T \circledcirc \lambda_3)$$

and

$$\sigma_{T\circledcirc D}^S = (\mu_S \circledcirc T \circledcirc D) \circ (S \circledcirc \psi_T^S \circledcirc D) \circ (\sigma_T^S \circledcirc \sigma_D^S) \circ (T \circledcirc \tau_D^T \otimes D) = \eta_S \circledcirc ((\mu_T \circledcirc \mu_D) \circ (T \circledcirc \lambda_2 \circledcirc D)),$$

satisfies the equalities (4) and (5). Then, the pair, $(S \circledcirc T \circledcirc D, \mu_{S\circledcirc T\circledcirc D})$ is the iterated weak crossed of $(S \circledcirc T, \mu_{S\circledcirc T})$ and $(S \circledcirc D, \mu_{S\circledcirc D})$ with associated product

$$\mu_{S\circledcirc T\circledcirc D} = (\mu_S \circledcirc T \circledcirc D) \circ (\mu_S \circledcirc \sigma_{T\circledcirc D}^S) \circ (S \circledcirc \psi_{T\circledcirc D}^S \circledcirc T \circledcirc D) =$$
$$(\mu_S \circledcirc \mu_T \circledcirc \mu_D) \circ (S \circledcirc ((\lambda_1 \circledcirc \lambda_2) \circ (T \circledcirc \lambda_3 \circledcirc T)) \circledcirc D).$$

In this case the preunits are units. The object $S \circledcirc T \circledcirc D$ is a monad with unit

$$\eta_{S\circledcirc T\circledcirc D} = \eta_S \circledcirc \eta_T \circledcirc \eta_D$$

because $S \circledcirc T$ and $S \circledcirc D$ are also monads with unit $\eta_{S\circledcirc T} = \eta_S \circledcirc \eta_T$ and $\eta_{S\circledcirc D} = \eta_S \circledcirc \eta_D$ respectively. Therefore, (73) and (74) holds and the morphism $\nu_{T\circledcirc S}$ defined in (75) is $\eta_{S\circledcirc T\circledcirc D}$.

On the other hand, the morphism ψ, introduced in (77), is

$$\psi = \eta_S \circledcirc \lambda_2.$$

Then, it is a distributive law of $(S \circledcirc T, \mu_{S\circledcirc T})$ over $(S \circledcirc D, \mu_{S\circledcirc D})$. As a consequence, in this case, the ψ-iterated product is the one defined in Theorem 2.4.

For example, if $\mathcal{C}$ is a strict monoidal category and A, B are monoids in $\mathcal{C}$ the twisted tensor product of algebras introduced in [11], [28] is an example weak crossed product associated to a wreath for the monad $S = A \otimes -$. In this case $T = B \otimes -$ and $\lambda = R \otimes -$ where $R : B \otimes A \to A \otimes B$ is the twisting morphism. Furthermore, the natural transformation $\lambda = R \otimes -$ is a distributive law of the monad $S = A \otimes -$ over the monad $T = B \otimes -$ if and only if $R : B \otimes A \to A \otimes B$ is a unital twisting morphism. Suppose that A, B and C are monoids, let

$$R_1 : B \otimes A \to A \otimes B, \ \ R_2 : C \otimes B \to B \otimes C, \ \ R_3 : C \otimes A \to A \otimes C,$$

unital twisting morphisms, and consider the monads $S = A\otimes -$, $T = B\otimes -$, $D = C\otimes -$, the induced quadruples $\mathbb{S}_T$, $\mathbb{S}_D$ and the twisting morphism $\tau_D^T = R_2\otimes -$. Then the iterated product defined in Theorem 2.1 of [17] is the one associated to the quadruple

$$\mathbb{S}_{T\circledcirc D} = (S, T\circledcirc D, \psi^S_{T\circledcirc D}, \sigma^S_{T\circledcirc D})$$

when we apply the functors in the unit object of the category.

Example 3.2. Given to monads S and T, the notion of weak distributive law of the monad S over the monad T was introduced by Ross Street in [27] as follows. It consists of a natural transformation

$$\lambda : T\circledcirc S \to S\circledcirc T$$

such that satisfies (89), (91) and

$$\lambda\circ(\eta_T\circledcirc S) = (\mu_S\circledcirc T)\circ(S\circledcirc(\lambda\circ(\eta_T\circledcirc\eta_S))), \tag{94}$$

$$\lambda\circ(T\circledcirc\eta_S) = (S\circledcirc\mu_T)\circ((\lambda\circ(\eta_T\circledcirc\eta_S))\circledcirc T). \tag{95}$$

In this definition the axioms (94) and (95) can be replaced for the identity [[27], Proposition 2.2]:

$$(S\circledcirc\mu_T)\circ((\lambda\circ(\eta_T\circledcirc S)\circledcirc T) = (\mu_S\circledcirc T)\circ(S\circledcirc(\lambda\circ(T\circledcirc\eta_S))). \tag{96}$$

For a weak distributive law, the weak wreath product of T over S with respect to λ was defined by Street in Definition 2.5 of [27] as

$$\mu_{S\circledcirc T} = (\mu_S\circledcirc\mu_T)\circ(S\circledcirc\lambda\circledcirc T).$$

The same set of axioms for monoids in category of modules over a commutative ring can be found in [10]. Then, the conditions used in [10] define a weak wreath product associated to monads induced by monoids.

It follows by (89) and (91) that $\mu_{S\circledcirc T}$ is an associative product but possibly without unity. In any case, if we take the quadruple

$$\mathbb{S}_T = (S, T, \psi^S_T = \lambda, \sigma^S_T = (S\circledcirc\mu_T)\circ((\lambda\circ(T\circledcirc\eta_S))\circledcirc T)),$$

we obtain that $\mathbb{S}_T$ satisfies (1), (4), (5) and (10). The associated idempotent defined in (2) is

$$\nabla_{S\circledcirc T} = (\mu_S\circledcirc T)\circ(S\circledcirc(\lambda\circ(T\circledcirc\eta_S))).$$

Then, the weak wreath product defined by the weak distributive law is the one induced by the quadruple $\mathbb{S}_T$. Therefore, every weak wreath product with respect to λ is a weak crossed product. In this setting the morphism $\nu_T = \nabla_{S\circledcirc T}\circ(\eta_S\circledcirc\eta_T)$ is a preunit and $S\times T$ is a monoid with unit $\eta_{S\times T} = p_{S\circledcirc T}\circ\nu_T$ (see also [[14], Example 3.16]).

Note that the equality (96) implies that

$$\sigma^S_T = (S\circledcirc\mu_T)\circ((\nabla_{S\circledcirc T}\circ(\eta_S\circledcirc T)\circledcirc T) = \nabla_{S\circledcirc T}\circ(\eta_S\circledcirc\mu_T) = \lambda\circ(\mu_T\circledcirc\eta_S). \tag{97}$$

Suppose that S, T and D are monads in $\mathcal{C}$ such that there exists tree weak distributive laws between them

$$\lambda_1 : T\circledcirc S\to S\circledcirc T,\quad \lambda_2 : D\circledcirc T\to T\circledcirc D,\quad \lambda_3 : D\circledcirc S\to S\circledcirc D,$$

satisfying the Yang-Baxter relation (93). Then, under these conditions we have two quadruples

$$\mathbb{S}_T = (S, T, \psi^S_T = \lambda_1, \sigma^S_T = (S\circledcirc\mu_T)\circ((\lambda_1\circ(T\circledcirc\eta_S))\circledcirc T)),$$

$$\mathbb{S}_D = (S, D, \psi^S_D = \lambda_3, \sigma^S_D = (S\circledcirc\mu_D)\circ((\lambda_3\circ(D\circledcirc\eta_S))\circledcirc D)),$$

satisfying (1), (4), (5) and (10). If we put $\Delta_{T\circledcirc D} = \nabla_{T\circledcirc D}$ we obtain a link morphism. Indeed, we have that (65) holds because

$$\begin{aligned}
&(S\circledcirc\nabla_{T\circledcirc D})\circ\psi^S_{T\circledcirc D}\\
&= (S\circledcirc\mu_T\circledcirc\mu_D)\circ(\lambda_1\circledcirc\lambda_2\circledcirc D)\circ(T\circledcirc\lambda_3\circledcirc\lambda_2)\circ((\lambda_2\circ(\eta_D\circledcirc T))\circledcirc\lambda_3\circledcirc\eta_T)\\
&= (S\circledcirc((\mu_T\circledcirc\mu_D)\circ(T\circledcirc\lambda_2\circledcirc D)\circ(\lambda_2\circledcirc T\circledcirc D)))\circ(\lambda_3\circledcirc T\circledcirc T\circledcirc D)\circ(D\circledcirc\lambda_1\circledcirc\lambda_2)\circ(\eta_D\circledcirc D\circledcirc\lambda_3\circledcirc\eta_T)\\
&= (S\circledcirc T\circledcirc\mu_D)\circ(S\circledcirc\lambda_2\circledcirc D)\circ(\lambda_3\circledcirc\nabla_{T\circledcirc D})\circ(D\circledcirc\lambda_1\circledcirc D)\circ(\eta_D\circledcirc T\circledcirc\lambda_3)\\
&= (S\circledcirc T\circledcirc\mu_D)\circ(((S\circledcirc\lambda_2)\circ(\lambda_3\circledcirc T)\circ(D\circledcirc\lambda_1))\circledcirc D)\circ(\eta_D\circledcirc T\circledcirc\lambda_3)
\end{aligned}$$

$= (\lambda_1 \circledcirc \mu_D) \circ (T \circledcirc \lambda_3 \circledcirc D) \circ ((\lambda_2 \circ (\eta_D \circledcirc T)) \circledcirc \lambda_3)$
$= \psi^S_{T\circledcirc D}$

In the last equalities, the first one follows by (89) for λ_3 and λ_2, the second one follows by (93), the third one follows by (91) for λ_2, the fourth one follows by

$$(T \circledcirc \mu_D) \circ (\lambda_2 \circledcirc D) \circ (D \circledcirc \nabla_{T\circledcirc D}) = (T \circledcirc \mu_D) \circ (\lambda_2 \circledcirc D), \tag{98}$$

the fifth one relies on (93) and the last one is a consequence of (89) for λ_3.

The equality (66) follows by

$$\begin{aligned}
&\nabla_{S\circledcirc T\circledcirc D} \circ (\lambda_1 \circledcirc D) \circ (T \circledcirc \lambda_3)\\
&= (((\mu_S \circledcirc \mu_T) \circ (S \circledcirc \lambda_1 \circledcirc T)) \circledcirc D) \circ (\lambda_1 \circledcirc ((\lambda_1 \circledcirc D) \circ (T \circledcirc \lambda_3) \circ (\lambda_2 \circledcirc S))) \circ (T \circledcirc \lambda_3 \circledcirc \eta_T \circledcirc \eta_S)\\
&= (((\mu_S \circledcirc \mu_T) \circ (S \circledcirc \lambda_1 \circledcirc T)) \circledcirc D) \circ (\lambda_1 \circledcirc ((S \circledcirc \lambda_2) \circ (\lambda_3 \circledcirc T) \circ (D \circledcirc \lambda_1))) \circ (T \circledcirc \lambda_3 \circledcirc \eta_T \circledcirc \eta_S)\\
&= (S \circledcirc \mu_T \circledcirc D) \circ (\lambda_1 \circledcirc \lambda_2) \circ (T \circledcirc \lambda_3 \circledcirc T) \circ (T \circledcirc D \circledcirc (\nabla_{S\circledcirc T} \circ (S \circledcirc \eta_T)))\\
&= (S \circledcirc \mu_T \circledcirc D) \circ (\lambda_1 \circledcirc T \circledcirc D) \circ (T \circledcirc (((S \circledcirc \lambda_2) \circ (\lambda_3 \circledcirc T) \circ (D \circledcirc \lambda_1)) \circ (D \circledcirc \eta_T \circledcirc S)))\\
&= (S \circledcirc \mu_T \circledcirc D) \circ (\lambda_1 \circledcirc T \circledcirc D) \circ (T \circledcirc (((\lambda_1 \circledcirc D) \circ (T \circledcirc \lambda_3) \circ (\lambda_2 \circledcirc S)) \circ (D \circledcirc \eta_T \circledcirc S)))\\
&= \psi^S_{T\circledcirc D}
\end{aligned}$$

where, the first and the sixth equalities follow by (89) for λ_1, the second and the fifth ones follow by (93), the third relies on (91) for λ_1 and λ_3 and, finally, the fourth one follows by (96).

On the other hand, if $\tau^T_D = \lambda_2$ we obtain that the condition (i) of Definition 2.3 holds by (93). Moreover, condition (ii) of the same Definition also holds because we have the following:

$$\begin{aligned}
&(\mu_S \circledcirc T \circledcirc D) \circ (S \circledcirc \sigma^S_T \circledcirc D) \circ (\psi^S_T \circledcirc \tau^T_D) \circledcirc (T \circledcirc \sigma^S_D \circledcirc T) \circ (\tau^T_D \circledcirc D \circledcirc T)\\
&= (S \circledcirc \mu_T \circledcirc D) \circ (\lambda_1 \circledcirc (\lambda_2 \circ (\mu_D \circledcirc T))) \circ (((T \circledcirc \lambda_3) \circ (\lambda_2 \circledcirc \eta_S)) \circledcirc D \circledcirc T)\\
&= (S \circledcirc ((\mu_T \circledcirc \mu_D) \circ (T \circledcirc \lambda_2 \circledcirc D) \circ (\lambda_2 \circledcirc T \circledcirc D))) \circ (((\lambda_3 \circledcirc T) \circ (D \circledcirc (\lambda_1 \circ (T \circledcirc \eta_S)))) \circledcirc \lambda_2)\\
&= (S \circledcirc T \circledcirc \mu_D) \circ (((S \circledcirc \lambda_2) \circ (\lambda_3 \circledcirc \mu_T) \circ (D \circledcirc (\lambda_1 \circ (T \circledcirc \eta_S)) \circledcirc T)) \circledcirc D) \circ (D \circledcirc T \circledcirc \lambda_2)\\
&= (S \circledcirc T \circledcirc \mu_D) \circ (((S \circledcirc \lambda_2) \circ (\lambda_3 \circledcirc T) \circ (D \circledcirc (\lambda_1 \circ (\mu_T \circledcirc \eta_S)))) \circledcirc D) \circ (D \circledcirc T \circledcirc \lambda_2)\\
&= (\lambda_1 \circledcirc \mu_D) \circ (T \circledcirc \lambda_3 \circledcirc D) \circ ((\lambda_2 \circ (D \circledcirc \mu_T)) \circledcirc \eta_S \circledcirc D) \circ (D \circledcirc T \circledcirc \lambda_2)\\
&= (((S \circledcirc \mu_T) \circ (\lambda_1 \circledcirc T)) \circledcirc D) \circ (T \circledcirc ((\lambda_1 \circledcirc \mu_D) \circ (T \circledcirc \lambda_3 \circledcirc D) \circ (\lambda_2 \circledcirc \eta_S \circledcirc D))) \circ (\lambda_2 \circledcirc \lambda_2)\\
&= (S \circledcirc \mu_T \circledcirc \mu_D) \circ (((\lambda_1 \circledcirc \lambda_2) \circ (T \circledcirc \lambda_3 \circledcirc T) \circ (\lambda_2 \circledcirc (\lambda_1 \circ (T \circledcirc \eta_S)))) \circledcirc D) \circ (D \circledcirc T \circledcirc \lambda_2)\\
&= (S \circledcirc ((\mu_T \circledcirc \mu_D) \circ (T \circledcirc \lambda_2 \circledcirc D) \circ (\lambda_2 \circledcirc T \circledcirc D))) \circ (\lambda_3 \circledcirc T \circledcirc T \circledcirc D) \circ (D \circledcirc (\lambda_1 \circ (T \circledcirc \mu_S)) \circledcirc T \circledcirc D)\\
&\circ (D \circledcirc T \circledcirc \eta_S \circledcirc (((\lambda_1 \circ (T \circledcirc \eta_S)) \circledcirc D) \circ \lambda_2))\\
&= (\mu_S \circledcirc ((\mu_T \circledcirc \mu_D) \circ (T \circledcirc \lambda_2 \circledcirc D) \circ (\lambda_2 \circledcirc T \circledcirc D))) \circ (S \circledcirc \lambda_3 \circledcirc T \circledcirc T \circledcirc D) \circ (\lambda_3 \circledcirc \lambda_1 \circledcirc T \circledcirc D)\\
&\circ (D \circledcirc (\lambda_1 \circ (T \circledcirc \eta_S)) \circledcirc (((\lambda_1 \circ (T \circledcirc \eta_S)) \circledcirc D) \circ \lambda_2))\\
&= (\mu_S \circledcirc T \circledcirc \mu_D) \circ (S \circledcirc S \circledcirc \lambda_2 \circledcirc D) \circ (S \circledcirc \lambda_3 \circledcirc T \circledcirc D) \circ (\lambda_3 \circledcirc (\lambda_1 \circ (\mu_T \circledcirc \eta_S))) \circledcirc D) \circ (D \circledcirc (\lambda_1 \circ (T \circledcirc \eta_S)) \circledcirc \lambda_2)\\
&= (\mu_S \circledcirc T \circledcirc D) \circ (S \circledcirc \psi^S_T \circledcirc D) \circ (S \circledcirc T \circledcirc \sigma^S_D) \circ (S \circledcirc \tau^T_D \circledcirc D) \circ (\psi^S_D \circledcirc T \circledcirc D) \circ (D \circledcirc \sigma^S_T \circledcirc D) \circ\\
&\circ (D \circledcirc T \circledcirc \tau^T_D).
\end{aligned}$$

In the last equalities, the fist one follows by (91) for λ_1 and

$$(S \circledcirc \mu_T) \circ (\lambda_1 \circledcirc T) \circ (T \circledcirc \nabla_{S\circledcirc T}) = (S \circledcirc \mu_T) \circ (\lambda_1 \circledcirc T),$$

the second one follows by (89) for λ_2 and (93), the third one follows by (91) for λ_2 and the fourth one is a consequence of (97) for λ_1. In the fifth one we used (93) and the sixth one relies on (91) for λ_2 and (89) for λ_1. The seventh one follows by (93), the eighth one follows by $\lambda_1 \circ (T \circledcirc \eta_S) = \nabla_{S\circledcirc T} \circ (\eta_S \circledcirc T)$ and (93). Finally, in the ninth one we used (91) for λ_1 and λ_3, the tenth one follows by (91) for λ_2 and (89) for λ_1 and the last one follows by (93).

Therefore, $\tau^T_D = \lambda_2$ is a twisting morphism between the quadruples $\mathbb{S}_T$ and $\mathbb{S}_D$.

If we put

$$\sigma^S_{T\circledcirc D} = (\mu_S \circledcirc T \circledcirc D) \circ (S \circledcirc \psi^S_T \circledcirc D) \circ (\sigma^S_T \circledcirc \sigma^S_D) \circ (T \circledcirc \tau^T_D \otimes D)$$

we obtain that

$$\sigma^S_{T\circledcirc D} = (\lambda_1 \circledcirc \mu_D) \circ (\mu_T \circledcirc \lambda_3 \circledcirc D) \circ (T \circledcirc \lambda_2 \circledcirc \eta_S \circledcirc D) \tag{99}$$

and $\sigma^S_{T\circledcirc D}$ satisfies (70), (71) and (72). Indeed: the equality (70) follows by

$$(\mu_T \circledcirc D) \circ (T \circledcirc \lambda_2) \circ (\nabla_{T\circledcirc D} \circledcirc T) = (\mu_T \circledcirc D) \circ (T \circledcirc \lambda_2) \tag{100}$$

and the proof for (71) is

$$\begin{aligned} &\sigma^{S}_{T\circledcirc D}\circ (T\circledcirc D\circledcirc \nabla_{T\circledcirc D})\\ =&(\lambda_1\circledcirc D)\circ(\mu_T\circledcirc(\lambda_3\circ(\mu_D\circledcirc\eta_S)))\circ(T\circledcirc\lambda_2\circledcirc D)\circ(T\circledcirc D\circledcirc\nabla_{T\circledcirc D})\\ =&(\lambda_1\circledcirc D)\circ(\mu_T\circledcirc(\lambda_3\circ(\mu_D\circledcirc\eta_S)))\circ(T\circledcirc\lambda_2\circledcirc D)\\ =&\sigma^{S}_{T\circledcirc D}\end{aligned}$$

where the first and the third equalities follows by

$$(S\circledcirc\mu_D)\circ((\lambda_3\circ(D\circledcirc\eta_S))\circledcirc D)=\lambda_3\circ(\mu_D\circledcirc\eta_S) \tag{101}$$

and the second one follows by (98).

Finally, by

$$\nabla_{T\circledcirc D}\circ(T\circledcirc\mu_D)=(T\circledcirc\mu_D)\circ(\nabla_{T\circledcirc D}\circledcirc D),$$
$$\nabla_{T\circledcirc D}\circ(\mu_T\circledcirc D)\circ(T\circledcirc\lambda_2)=(\mu_T\circledcirc D)\circ(T\circledcirc\lambda_2)$$

and (93) we obtain (72).

As a consequence, by Lemma 2.2 and Theorem 2.4, the quadruple

$$\mathbb{S}_{T\circledcirc D}=(S,T\circledcirc D,\Gamma^{S}_{T\circledcirc D},\sigma^{S}_{T\circledcirc D}),$$

where

$$\Gamma^{S}_{T\circledcirc D}=(\lambda_1\circledcirc D)\circ(T\circledcirc\lambda_3)\circ(\nabla_{T\circledcirc D}\circledcirc S),$$

satisfies the equalities (4) and (5). Then, $(S\circledcirc T\circledcirc D,\mu_{S\circledcirc T\circledcirc D})$ is the iterated weak crossed product of $(S\circledcirc T,\mu_{S\circledcirc T})$ and $(S\circledcirc D,\mu_{S\circledcirc D})$ with associated product

$$\mu_{S\circledcirc T\circledcirc D}=(\mu_S\circledcirc T\circledcirc D)\circ(\mu_S\circledcirc\sigma^{S}_{T\circledcirc D})\circ(S\circledcirc\Gamma^{S}_{T\circledcirc D}\circledcirc T\circledcirc D),$$

and equivalently

$$\begin{aligned}\mu_{S\circledcirc T\circledcirc D}&=(\mu_S\circledcirc\mu_T\circledcirc\mu_D)\circ(S\circledcirc((\lambda_1\circledcirc\lambda_2)\circ(T\circledcirc\lambda_3\circledcirc T)\circ(\nabla_{T\circledcirc D}\circledcirc\nabla_{S\circledcirc T}))\circledcirc D)\\ &=(\mu_S\circledcirc\mu_T\circledcirc\mu_D)\circ(S\circledcirc((\lambda_1\circledcirc\lambda_2)\circ(T\circledcirc\lambda_3\circledcirc T)\circ(T\circledcirc D\circledcirc\nabla_{S\circledcirc T}))\circledcirc D).\end{aligned} \tag{102}$$

Also, we have the preunit conditions of Theorem 2.6. Indeed, the proof for (73) is the following:

$$\begin{aligned} &(\mu_S\circledcirc T\circledcirc D)\circ(S\circledcirc\sigma^{S}_{T}\circledcirc D)\circ(\lambda_1\circledcirc\lambda_2)\circ(T\circledcirc\lambda_3\circledcirc T)\circ(T\circledcirc D\circledcirc\nu_T)\\ =&(S\circledcirc\mu_T\circledcirc D)\circ(\lambda_1\circledcirc\lambda_2)\circ(T\circledcirc\lambda_3\circledcirc T)\circ(T\circledcirc D\circledcirc\nu_T)\\ =&(S\circledcirc\mu_T\circledcirc D)\circ(\lambda_1\circledcirc T\circledcirc D)\circ(T\circledcirc((S\circledcirc\lambda_2)\circ(\lambda_3\circledcirc T)\circ(D\circledcirc\lambda_1)))\circ(T\circledcirc D\circledcirc\eta_T\circledcirc\eta_S)\\ =&(S\circledcirc\mu_T\circledcirc D)\circ(\lambda_1\circledcirc T\circledcirc D)\circ(T\circledcirc((\lambda_1\circledcirc D)\circ(T\circledcirc\lambda_3)\circ(\lambda_2\circledcirc S)))\circ(T\circledcirc D\circledcirc\eta_T\circledcirc\eta_S)\\ =&\nabla_{S\circledcirc T\circledcirc D}\circ(\eta_S\circledcirc T\circledcirc D)\end{aligned}$$

where the first equality follows by (96) and (91) for λ_1, the second one follows by the definition of $\nabla_{S\circledcirc T}$, the third one relies on (93) and the last one is a consequence (89) for λ_1.

Finally, (74) follows by

$$\begin{aligned} &(\mu_S\circledcirc T\circledcirc D)\circ(S\circledcirc\lambda_1\circledcirc D)\circ(S\circledcirc T\circledcirc\sigma^{S}_{D})\circ(S\circledcirc\lambda_2\circledcirc D)\circ(\nu_D\circledcirc T\circledcirc D)\\ =&(S\circledcirc T\circledcirc\mu_D)\circ(((S\circledcirc\lambda_2)\circ(\lambda_3\circledcirc T)\circ(D\circledcirc\lambda_1))\circledcirc D)\circ(\eta_D\circledcirc T\circledcirc\eta_S\circledcirc D)\\ =&(\lambda_1\circledcirc\mu_D)\circ(T\circledcirc\lambda_3\circledcirc D)\circ(\lambda_2\circledcirc S\circledcirc D)\circ(\eta_D\circledcirc T\circledcirc\eta_S\circledcirc D)\\ =&\nabla_{S\circledcirc T\circledcirc D}\circ(\eta_S\circledcirc T\circledcirc D)\end{aligned}$$

where the first equality is a consequence of (93) and

$$(\mu_S\circledcirc D)\circ(S\circledcirc\lambda_3)\circ(\nabla_{S\circledcirc D}\circledcirc S)=(\mu_S\circledcirc D)\circ(S\circledcirc\lambda_3),$$

the second one of (93) and the last one of (101).

Therefore,

$$\nu_{T\circledcirc D}=\nabla_{S\circledcirc T\circledcirc D}\circ(\mu_S\circledcirc T\circledcirc D)\circ(S\circledcirc\lambda_1\circledcirc D)\circ(\nu_T\circledcirc\nu_D)$$

is a preunit for $\mu_{S\circledcirc T\circledcirc D}$ and we have that

$$\nu_{T\circledcirc D}=(\lambda_1\circledcirc D)\circ(T\circledcirc\lambda_3)\circ(\lambda_2\circledcirc S)\circ(\eta_D\circledcirc\eta_T\circledcirc\eta_S).$$

In this case the morphism ψ, introduced in (77), is

$$\psi=(S\circledcirc\lambda_2)\circ(\lambda_3\circledcirc T)\circ(D\circledcirc(\lambda_1\circ(T\circledcirc\eta_S))).$$

It is not a distributive law of $(S \circledcirc T, \mu_{S\circledcirc T})$ over $(S \circledcirc D, \mu_{S\circledcirc D})$, because the conditions (56), and (57) do not hold. In any case, it is a weak distributive law. Note that, using (89), (91), and (93) it is easy to show that (52), (53), (54), and (55) hold. Moreover, using that

$$\gamma_S = (\mu_S \circledcirc \lambda_2) \circ (S \circledcirc \lambda_3 \circledcirc T) \circ ((\lambda_3 \circ (\eta_D \circledcirc S)) \circledcirc (\lambda_1 \circ (T \circledcirc \eta_S)))$$

we obtain the equality

$$\gamma_S \circ \nu_T = (S \circledcirc \lambda_2) \circ (\lambda_3 \circledcirc T) \circ (D \circledcirc \lambda_1) \circ (\eta_D \circledcirc \eta_T \circledcirc \eta_S),$$

and then, applying (89), (91), (93) and (96), we can prove (58) and (59). As a consequence, the ψ-iterated product is the one defined in Theorem 2.4.

Example 3.3. In this example we will show that the iteration process proposed recently by Dăuş and Panaite in [13] for Brzeziński's crossed products, is a particular case of the weak iterated products defined in this paper. First we recall from [9] the construction of Brzeziński's crossed product in a strict monoidal category: Let (A, η_A, μ_A) be a monoid and V an object equipped with a distinguished morphism $\eta_V : K \to V$. Then the object $A \otimes V$ is a monoid with unit $\eta_A \otimes \eta_V$ and whose product has the property $\mu_{A\otimes V} \circ (A \otimes \eta_V \otimes A \otimes V) = \mu_A \otimes V$, if and only if there exists two morphisms $\psi_V^A : V \otimes A \to A \otimes V$, $\sigma_V^A : V \otimes V \to A \otimes V$ satisfying (1), the twisted condition (4), the cocycle condition (5) and

$$\psi_V^A \circ (\eta_V \otimes A) = A \otimes \eta_V, \tag{103}$$

$$\psi_V^A \circ (V \otimes \eta_A) = \eta_A \otimes V, \tag{104}$$

$$\sigma_V^A \circ (\eta_V \otimes V) = \sigma_V^A \circ (V \otimes \eta_V) = \eta_A \otimes V. \tag{105}$$

If this is the case, the product of $A \otimes V$ is the one defined in (11). Note that Brzeziński's crossed products are examples of weak crossed products where the associated idempotent is the identity, that is, $\nabla_{A\otimes V} = id_{A\otimes V}$. Also, in this case the preunit $\nu = \eta_A \otimes \eta_V$ is a unit.

Given two Brzeziński's crossed products for $A\otimes V$ and $A\otimes W$, in [13] a new crossed product is defined in $A \otimes V \otimes W$ if there exists a morphism $\tau_W^V : W \otimes V \to V \otimes W$ satisfying the condition (i) of Definition 2.3, (79), (80) and

$$\tau_W^V \circ (\eta_W \otimes V) = V \otimes \eta_W, \tag{106}$$

$$\tau_W^V \circ (W \otimes \eta_V) = \eta_V \otimes W. \tag{107}$$

In this case, $\psi_{V\otimes W}^A = (\psi_V^A \otimes W) \circ (V \otimes \psi_W^A)$, $\sigma_{V\otimes W}^A$ is defined as in (69) and $\eta_{V\otimes W} = \eta_V \otimes \eta_W$.

Under these conditions, by Remark 2.9, we have that τ_W^V is a twisting morphism and, if we consider the link morphism $\Delta_{V\otimes W} = id_{V\otimes W}$, we obtain that the iterated crossed product proposed in [13] is a particular instance of the iterated weak crossed product introduced in Theorem 2.4. Moreover, in this setting, if the equality (ii) of Definition 2.3 holds, composing with $W \otimes V \otimes \eta_W \otimes V$ in both sides we obtain (80), and composing with $W \otimes \eta_V \otimes W \otimes V$ we obtain (79).

Note that, in this case, we also have that the morphism ψ, introduced in (77), is a distributive law of $(A \otimes V, \mu_{A\otimes V})$ over $(A \otimes W, \mu_{A\otimes W})$. In this case, the ψ-iterated product is the one defined in Theorem 2.4.

4. A different characterization of the iterated weak crossed product

In this section we obtain a new characterization of the iteration process following Theorem 1.4 of [15]. This theorem asserts the following:

Theorem 4.1. *Let T and B be a monoids in $\mathcal{C}$. Then the following are equivalent:*

(i) *There exist a weak crossed product $(B\otimes W, \mu_{B\otimes W})$ with preunit ν and an isomorphism of monoids $\omega : B \times W \to T$.*

(ii) *There exist an algebra B, an object W, morphisms*

$$i_B : B \to T, \quad i_W : W \to T, \quad \nabla_{B\otimes W} : B \otimes W \to B \otimes W, \quad \omega : B \times W \to T$$

such that i_B is a monoid morphism, $\nabla_{B\otimes W}$ is an idempotent morphism of left B-modules for the action $\varphi_{B\otimes W} = \mu_B \otimes W$, and ω is an isomorphism such that

$$\omega \circ p_{B\otimes W} = \mu_T \circ (i_B \otimes i_W)$$

where $B \times W$ is the image of $\nabla_{B\otimes W}$ and $p_{B\otimes W}$ is the associated projection.

Theorem 4.2. *Let $\mathbb{A}_V = (A, V, \psi_V^A, \sigma_V^A)$, $\mathbb{A}_W = (A, W, \psi_W^A, \sigma_W^A)$ be two quadruples satisfying (4) and (5) with a link morphism $\Delta_{V\otimes W} : V \otimes W \to V \otimes W$ and with a twisting morphism $\tau_W^V : W \otimes V \to V \otimes W$ between them. Let $(A \otimes V, \mu_{A\otimes V})$ and $(A \otimes W, \mu_{A\otimes W})$ be the weak crossed products associated to $\mathbb{A}_V$ and $\mathbb{A}_W$ and suppose that $\nu_V : K \to A \otimes V$ and $\nu_W : K \to A \otimes W$ are preunits for $\mu_{A\otimes V}$ and $\mu_{A\otimes W}$. Assume that the morphism $\sigma_{V\otimes W}^A$, defined in (69), satisfies (70), (71), (72) and assume also that the equalities (73) and (74) hold.*

(i) *Let $i_{A\times V} : A \times V \to A \times (V \otimes W)$ be the morphism defined by*

$$i_{A\times V} = p_{A\otimes V\otimes W} \circ (\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (i_{A\otimes V} \otimes \upsilon_W),$$

where $A \times (V \otimes W)$ is the image of the idempotent morphism $\nabla_{A\otimes V\otimes W}$ introduced in Definition 2.1 and $p_{A\otimes V\otimes W}$ its associated projection.

If the equality

$$\nabla_{A\otimes V\otimes W} \circ (((\mu_A \otimes V) \circ (A \otimes \psi_V^A) \circ (\sigma_V^A \otimes A)) \otimes W) \circ (V \otimes V \otimes \nu_W) \tag{108}$$
$$= \nabla_{A\otimes V\otimes W} \circ (((\mu_A \otimes V) \circ (A \otimes \sigma_V^A)) \otimes W) \circ (\psi_V^A \otimes \tau_W^V) \circ (V \otimes \nu_W \otimes V),$$

holds, $i_{A\times V}$ is a monoid morphism.

(ii) *If $A \times V$, $p_{A\times V}$ and $i_{A\times V}$ are the image, the projection and the injection associated a $\nabla_{A\otimes V}$, the morphism $\nabla_{(A\times V)\otimes W} : (A \times V) \otimes W \to (A \times V) \otimes W$ defined by*

$$\nabla_{(A\times V)\otimes W} = (p_{A\otimes V} \otimes W) \circ \nabla_{A\otimes V\otimes W} \circ (i_{A\otimes V} \otimes W),$$

is idempotent. Moreover, if the following identity holds

$$\nabla_{A\otimes V\otimes W} \circ (\sigma_V^A \otimes W) = (((\mu_A \otimes V) \circ (A \otimes \psi_V^A)) \otimes W) \circ (\sigma_V^A \otimes \psi_W^A) \circ (V \otimes \Delta_{V\otimes W} \otimes \eta_A), \tag{109}$$

$\nabla_{(A\times V)\otimes W}$ is a morphism of left $A \times V$-modules for $\varphi_{(A\times V)\otimes W} = \mu_{A\times V} \otimes W$.

(iii) *The morphism $\omega : (A \times V) \times W \to A \times (V \otimes W)$ defined by*

$$\omega = p_{A\otimes V\otimes W} \circ (i_{A\otimes V} \otimes W) \circ i_{(A\times V)\otimes W},$$

where $i_{(A\times V)\otimes W}$ is the injection associated to $\nabla_{(A\times V)\otimes W}$, is an isomorphism. Moreover, if the equality

$$\nabla_{A\otimes V\otimes W} \circ (\psi_V^A \otimes W) \circ (V \otimes \sigma_W^A) = (\psi_V^A \otimes W) \circ (V \otimes \sigma_W^A) \circ (\Delta_{V\otimes W} \otimes W) \tag{110}$$

holds, then

$$\omega \circ p_{(A\times V)\otimes W} = \mu_{A\times(V\otimes W)} \circ (i_{A\times V} \otimes i_W)$$

for

$$i_W = p_{A\otimes V\otimes W} \circ (\nu_V \otimes W).$$

Therefore, if (108), (109) and (110) hold, $A \times (V \otimes W)$ and $(A \times V) \times W$ are isomorphic as monoids.

Proof. The proof for (i) is the following:

$$\begin{aligned}
&\mu_{A\times(V\otimes W)} \circ (i_{A\times V} \otimes i_{A\times V}) \\
&= p_{A\otimes V\otimes W} \circ (\mu_A \otimes V \otimes W) \circ (\mu_A \otimes \sigma_{V\otimes W}^A) \circ (A \otimes \Gamma_{V\otimes W}^A \otimes V \otimes W) \\
&\circ (((\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (i_{A\otimes V} \otimes \nu_W)) \otimes ((\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (i_{A\otimes V} \otimes \nu_W))) \\
&= p_{A\otimes V\otimes W} \circ (\mu_A \otimes V \otimes W) \circ (\mu_A \otimes ((\mu_A \otimes V \otimes W) \circ (A \otimes \psi_V^A \otimes W) \circ (\sigma_V^A \otimes \sigma_W^A))) \circ (\mu_A \otimes \psi_V^A \otimes V \otimes W \otimes W) \\
&\circ (A \otimes \psi_V^A \otimes ((A \otimes \tau_W^V) \circ (\psi_W^A \otimes V) \circ (W \otimes \psi_V^A)) \otimes W) \circ (A \otimes V \otimes \psi_W^A \otimes V \otimes A \otimes W) \circ (((\mu_A \otimes \Delta_{V\otimes W}) \\
&\circ (A \otimes \psi_V^A \otimes W) \circ (i_{A\otimes V} \otimes \nu_W)) \otimes i_{A\otimes W} \otimes \nu_W)
\end{aligned}$$

$$
\begin{aligned}
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes W)\\
&\quad\circ(\mu_A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\psi_V^A))\otimes\sigma_W^A)\circ(A\otimes A\otimes V\otimes V\otimes\psi_W^A\otimes W)\\
&\quad\circ(\mu_A\otimes A\otimes V\otimes\tau_W^V\otimes\nu_W)\circ(A\otimes A\otimes((\psi_V^A\otimes W)\circ(V\otimes\psi_W^A)\circ(\Delta_{V\otimes W}\otimes A))\otimes V)\\
&\quad\circ(((A\otimes\psi_V^A\otimes W)\circ(i_{A\otimes V}\otimes\nu_W))\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes W)\\
&\quad\circ(\mu_A\otimes((\mu_A\otimes V)\circ(A\otimes\psi_V^A)\circ(\sigma_V^A\otimes V))\otimes\sigma_W^A)\circ(A\otimes A\otimes V\otimes V\otimes\psi_W^A\otimes W)\\
&\quad\circ(\mu_A\otimes A\otimes V\otimes\tau_W^V\otimes\nu_W)\circ(A\otimes A\otimes(\nabla_{A\otimes V\otimes W}\circ(\psi_V^A\otimes W)\circ(V\otimes\psi_W^A))\otimes V)\\
&\quad\circ(((A\otimes\psi_V^A\otimes W)\circ(i_{A\otimes V}\otimes\nu_W))\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes W)\\
&\quad(\mu_A\otimes\sigma_V^A\otimes((\mu_A\otimes W)\circ(A\otimes\sigma_W^A)\circ(\psi_W^A\otimes W)\circ(W\otimes\nu_W)))\circ(A\otimes\mu_A\otimes V\otimes\tau_W^V)\\
&\quad\circ(A\otimes A\otimes(\nabla_{A\otimes V\otimes W}\circ(\psi_V^A\otimes W)\circ(V\otimes\psi_W^A))\otimes V)\circ(((A\otimes\psi_V^A\otimes W)\circ(i_{A\otimes V}\otimes\nu_W))\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes\sigma_V^A\otimes(\psi_W^A\circ(W\otimes\eta_A)))\\
&\quad\circ(A\otimes A\otimes V\otimes\tau_W^V)\circ(A\otimes(\nabla_{A\otimes V\otimes W}\circ(A\otimes\psi_V^A\otimes W)\circ(V\otimes((\mu_A\otimes W)\circ(A\otimes\psi_W^A)\circ(\nu_W\otimes A))\otimes V)\\
&\quad\circ(i_{A\otimes V}\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A)))\\
&\quad\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma_V^A\otimes W)\circ(A\otimes V\otimes\tau_W^V)\circ(\nabla_{A\otimes V\otimes W}\otimes V)\\
&\quad\circ(\mu_A\otimes V\otimes W\otimes V)\circ(A\otimes\psi_V^A\otimes W\otimes V)\circ(i_{A\otimes V}\otimes((\beta_{\nu_W}\otimes V)\circ i_{A\otimes V}))\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A)))\\
&\quad\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\sigma_V^A\otimes W)\circ(\psi_V^A\otimes\tau_W^V)\circ(V\otimes\sigma_W^A\otimes V)\circ(\tau_W^V\otimes W\otimes V)))\\
&\quad\circ(A\otimes\nu_W\otimes V\otimes W\otimes V)\circ(\mu_A\otimes V\otimes W\otimes V)\circ(A\otimes\psi_V^A\otimes W\otimes V)\circ(i_{A\otimes V}\otimes((\beta_{\nu_W}\otimes V)\circ i_{A\otimes V}))\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes((\mu_A\otimes V)\circ(A\otimes\psi_V^A))\otimes W)\circ(\mu_A\otimes A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A)))\\
&\quad(A\otimes((\mu_A\otimes((\psi_V^A\otimes W)\circ(V\otimes\sigma_W^A)\circ(\tau_W^V\otimes W)))\circ(A\otimes\psi_W^A\otimes V\otimes W)\circ(\nu_W\otimes\sigma_V^A\otimes W)\circ(V\otimes\tau_W^V)))\\
&\quad\circ(\mu_A\otimes V\otimes W\otimes V)\circ(A\otimes\psi_V^A\otimes W\otimes V)\circ(i_{A\otimes V}\otimes((\beta_{\nu_W}\otimes V)\circ i_{A\otimes V}))\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes V\otimes(\nabla_{A\otimes W}\circ\sigma_W^A))\\
&\quad\circ(A\otimes((\mu_A\otimes\tau_W^V)\circ(A\otimes\psi_W^A\otimes V)\circ(\nu_W\otimes\sigma_V^A))\otimes W)\circ(A\otimes V\otimes\tau_W^V)\\
&\quad\circ(\mu_A\otimes V\otimes W\otimes V)\circ(A\otimes\psi_V^A\otimes W\otimes V)\circ(i_{A\otimes V}\otimes((\beta_{\nu_W}\otimes V)\circ i_{A\otimes V}))\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes V\otimes\sigma_W^A)\\
&\quad\circ(A\otimes((\mu_A\otimes\tau_W^V)\circ(A\otimes\beta_{\nu_W}\otimes V)\circ(A\otimes\sigma_V^A))\otimes W)\circ(A\otimes A\otimes V\otimes\tau_W^V)\\
&\quad\circ(A\otimes\psi_V^A\otimes W\otimes V)\circ(i_{A\otimes V}\otimes((\beta_{\nu_W}\otimes V)\circ i_{A\otimes V}))\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\mu_A\otimes V\otimes W)\\
&\quad\circ(A\otimes A\otimes A\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes\sigma_W^A)\circ(A\otimes\tau_W^V\otimes W)\circ(\nu_W\otimes V\otimes W)))\\
&\quad\circ(A\otimes\mu_A\otimes\sigma_V^A\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes\tau_W^V)\circ(A\otimes\psi_V^A\otimes\nu_W\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\mu_A\otimes V\otimes W)\\
&\quad\circ(A\otimes A\otimes A\otimes(\psi_{V\otimes W}^A\circ(V\otimes W\otimes\eta_A)))\\
&\quad\circ(A\otimes\mu_A\otimes\sigma_V^A\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes\tau_W^V)\circ(A\otimes\psi_V^A\otimes\nu_W\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\\
&\quad(A\otimes(\nabla_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\sigma_V^A\otimes W)\circ(\psi_V^A\otimes\tau_W^V)\circ(V\otimes\nu_W\otimes V)))\\
&\quad\circ(\mu_A\otimes V\otimes V)\circ(A\otimes\psi_V^A\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V})\\
&= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\\
&\quad(A\otimes(\nabla_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\sigma_V^A\otimes\nu_W)))\\
&\quad\circ(\mu_A\otimes V\otimes V)\circ(A\otimes\psi_V^A\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V})\\
&= i_{A\times V}\circ\mu_{A\times V}.
\end{aligned}
$$

The first equality follows because $\mu_{A\otimes V\otimes W}$ is normalized for $\nabla_{A\otimes V\otimes W}$, the second one relies on (1) for $\mathbb{A}_W$ and $\mathbb{A}_V$, and the third one follows by (i) of Definition 2.3 and the associativity of μ_A. In the fourth one we applied (66) and (4) for $\mathbb{A}_V$. The fifth one follows by (1) for $\mathbb{A}_V$ and the associativity of μ_A; the sixth one follows by the left linearity for $\nabla_{A\otimes V\otimes W}$, (17) for ν_W and (1) for $\mathbb{A}_V$; the seventh one follows by (19) for ν_W, the left linearity for $\nabla_{A\otimes V\otimes W}$ and the associativity of μ_A. The eighth one relies on (74) and the associativity of μ_A, the ninth one is a consequence of (ii) of Definition 2.3 and the associativity of μ_A, and the tenth one follows by (1) for $\mathbb{A}_V$. In the eleventh one we used (19) for ν_W, (19) for ν_W and (10) for σ_W^A. The twelfth one follows by (1) for $\mathbb{A}_V$ and the associativity of μ_A; the thirteenth one follows by (74) and the fourteenth one follows by the left linearity for $\nabla_{A\otimes V\otimes W}$. The fifteenth one is a consequence of (108) and the last one follows by the associativity of μ_A, the left linearity for $\nabla_{A\otimes V\otimes W}$ and by (3) for $\mathbb{A}_V$.

Therefore, $i_{A\times V}$ is multiplicative and, by (22), we have

$$\begin{aligned} i_{A\times V}\circ\eta_{A\times V} &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ((\nabla_{A\otimes V}\circ\nu_V)\otimes\nu_W)\\ &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\nu_V\otimes\nu_W) = \eta_{A\times(V\otimes W)}.\end{aligned}$$

(ii) The morphism $\nabla_{(A\times V)\otimes W} = (p_{A\otimes V}\otimes W)\circ\nabla_{A\otimes V\otimes W}\circ(i_{A\otimes V}\otimes W)$, is idempotent because

$$\begin{aligned} &\nabla_{(A\times V)\otimes W}\circ\nabla_{(A\times V)\otimes W}\\ &= (p_{A\otimes V}\otimes W)\circ\nabla_{A\otimes V\otimes W}\circ((\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\psi_V^A))\otimes W)\circ(A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A)))\\ &\quad\circ(A\otimes\Delta_{V\otimes W})\circ(i_{A\otimes V}\otimes W)\\ &= (p_{A\otimes V}\otimes W)\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A)))\circ(A\otimes((A\otimes\Delta_{V\otimes W})\\ &\quad\circ(\Gamma_{V\otimes W}^A\circ(V\otimes W\otimes\eta_A))))\circ(i_{A\otimes V}\otimes W)\\ &= (p_{A\otimes V}\otimes W)\circ(\mu_A\otimes V\otimes W)\circ(A\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\psi_V^A\otimes\psi_W^A)\circ(V\otimes\psi_W^A\otimes A)\\ &\quad\circ(V\otimes W\otimes\eta_A\otimes\eta_A)))\circ(A\otimes\Delta_{V\otimes W})\circ(i_{A\otimes V}\otimes W)\\ &= \nabla_{(A\times V)\otimes W},\end{aligned}$$

where the first equality follows by definition, the second one follows by (3), the third one relies on (65), and the las one follows by (1) for $\mathbb{A}_V$ and $\mathbb{A}_W$.

On the other hand,

$$\begin{aligned} &\nabla_{(A\times V)\otimes W}\circ\varphi_{(A\times V)\otimes W}\\ &= (p_{A\otimes V}\otimes W)\circ\nabla_{A\otimes V\otimes W}\circ((\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(\mu_A\otimes\sigma_V^A)\circ(A\otimes\psi_V^A\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V}))\otimes W)\\ &= (p_{A\otimes V}\otimes W)\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes(\nabla_{A\otimes V\otimes W}\circ(\sigma_V^A\otimes W)))\circ(((A\otimes\psi_V^A\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V}))\otimes W)\\ &= (p_{A\otimes V}\otimes W)\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\sigma_V^A\otimes(\psi_W^A\circ(W\otimes\eta_A)))\\ &\quad\circ(V\otimes\Delta_{V\otimes W})))\circ(((A\otimes\psi_V^A\otimes V)\circ(i_{A\otimes V}\otimes i_{A\otimes V}))\otimes W)\\ &= (p_{A\otimes V}\otimes W)\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes((\mu_A\otimes V)\circ(A\otimes\sigma_V^A)\circ(\psi_V^A\otimes V)\circ(V\otimes\psi_V^A))\otimes W)\\ &\quad\circ(A\otimes\psi_V^A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A)))\circ(A\otimes V\otimes A\otimes\Delta_{V\otimes W})\circ(i_{A\otimes V}\otimes i_{A\otimes V}\otimes W)\\ &= \varphi_{(A\times V)\otimes W}\circ(A\times V\otimes\nabla_{(A\times V)\otimes W})\end{aligned}$$

where the first equality follows by definition, the second one follows by the left linearity of $\nabla_{A\otimes V}$ and (10), the third one relies on (109) and the fifth one is a consequence of (3), (1) for $\mathbb{A}_V$ and the associativity of μ_A.

Finally, we will prove (iii). The morphism $\omega = p_{A\otimes V\otimes W}\circ(i_{A\otimes V}\otimes W)\circ i_{(A\times V)\otimes W}$ is an isomorphism with inverse

$$\omega^{-1} = p_{(A\times V)\otimes W}\circ(p_{A\otimes V}\otimes W)\circ i_{A\otimes V\otimes W}$$

because

$$\omega^{-1}\circ\omega = p_{(A\times V)\otimes W}\circ\nabla_{(A\times V)\otimes W}\circ i_{(A\times V)\otimes W} = id_{(A\times V)\otimes W}$$

and, by (3), (66) and the left linearity of $\nabla_{A\otimes V\otimes W}$, we have

$$\begin{aligned} &\omega\circ\omega^{-1}\\ &= p_{A\otimes V\otimes W}\circ((\nabla_{A\otimes V}\circ(\mu_A\otimes V)\circ(A\otimes\psi_V^A))\otimes W)\circ(A\otimes V\otimes\psi_W^A)\circ(A\otimes\Delta_{V\otimes W}\otimes\eta_A)\circ(\nabla_{A\otimes V}\otimes W)\\ &\quad\circ i_{A\otimes V\otimes W}\\ &= p_{A\otimes V\otimes W}\circ(((\mu_A\otimes V)\circ(A\otimes\psi_V^A))\otimes W)\circ(A\otimes V\otimes\psi_W^A)\circ(A\otimes\Delta_{V\otimes W}\otimes\eta_A)\circ(\nabla_{A\otimes V}\otimes W)\\ &\quad\circ i_{A\otimes V\otimes W}\\ &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\nabla_{A\otimes V\otimes W})\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes((\psi_W^A\circ(W\otimes\eta_A))))\circ(\nabla_{A\otimes V}\otimes W)\\ &\quad\circ i_{A\otimes V\otimes W}\\ &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes((\psi_W^A\circ(W\otimes\eta_A))))\circ(\nabla_{A\otimes V}\otimes W)\circ i_{A\otimes V\otimes W}\\ &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes((\psi_W^A\circ(W\otimes\eta_A))))\circ i_{A\otimes V\otimes W}\\ &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\nabla_{A\otimes V\otimes W})\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes((\psi_W^A\circ(W\otimes\eta_A))))\circ i_{A\otimes V\otimes W}\\ &= p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(A\otimes V\otimes((\psi_W^A\circ(W\otimes\eta_A))))\circ(A\otimes\Delta_{V\otimes W})\circ i_{A\otimes V\otimes W}\\ &= id_{A\times(V\otimes W)}.\end{aligned}$$

Moreover, if (110) holds, we have the following:

$$
\begin{aligned}
&\mu_{A\times(V\otimes W)}\circ(i_{A\times V}\otimes i_W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes\psi_V^A\otimes W)\circ(A\otimes A\otimes V\otimes\sigma_W^A)\\
&\quad\circ(\mu_A\otimes((\mu_A\otimes V\otimes W)\circ(A\otimes\sigma_V^A\otimes W)\circ(\psi_V^A\otimes\tau_W^V)\circ(V\otimes\psi_W^A\otimes V)\circ(\Delta_{V\otimes W}\otimes\nu_V))\otimes W)\\
&\quad\circ(A\otimes\psi_V^A\otimes W\otimes W)\circ(i_{A\otimes V}\otimes\nu_W\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\mu_A\otimes V\otimes W)\circ(A\otimes A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes\psi_V^A\otimes\sigma_W^A)\\
&\quad\circ(A\otimes A\otimes V\otimes(\psi_W^A\circ(W\otimes\eta_A))\otimes W)\circ(A\otimes A\otimes\Delta_{V\otimes W}\otimes W)\circ(A\otimes\psi_V^A\otimes W\otimes W)\circ(i_{A\otimes V}\otimes\nu_W\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma_W^A)\circ((\psi_W^A\circ(W\otimes\eta_A))\otimes W)\\
&\quad\circ(A\otimes A\otimes\Delta_{V\otimes W}\otimes W)\circ(A\otimes\psi_V^A\otimes W\otimes W)\circ(i_{A\otimes V}\otimes\nu_W\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(\mu_A\otimes V\otimes((\mu_A\otimes W)\circ(A\otimes\sigma_W^A)\circ(\nabla_{A\otimes W}\circ(\eta_A\otimes W))))\\
&\quad\circ(A\otimes A\otimes\Delta_{V\otimes W}\otimes W)\circ(A\otimes\psi_V^A\otimes W\otimes W)\circ(i_{A\otimes V}\otimes\nu_W\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes((\psi_V^A\otimes W)\circ(V\otimes\sigma_W^A)\circ(\Delta_{V\otimes W}\otimes W)))\circ(A\otimes\psi_V^A\otimes W\otimes W)\\
&\quad\circ(i_{A\otimes V}\otimes\nu_W\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(\mu_A\otimes(\nabla_{A\otimes V\otimes W}\circ(\psi_V^A\otimes W)\circ(V\otimes\sigma_W^A)))\circ(A\otimes\psi_V^A\otimes W\otimes W)\\
&\quad\circ(i_{A\otimes V}\otimes\nu_W\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes\psi_V^A\otimes W)\circ(i_{A\otimes V}\otimes((\mu_A\otimes W)\circ(A\otimes\sigma_W^A)\circ(\nu_W\otimes W)))\\
&=p_{A\otimes V\otimes W}\circ(\mu_A\otimes V\otimes W)\circ(A\otimes(\nabla_{A\otimes V\otimes W}\circ(\psi_V^A\otimes W)\circ(V\otimes(\psi_W^A\circ(W\otimes\eta_A)))))\circ(i_{A\otimes V}\otimes W)\\
&=p_{A\otimes V\otimes W}\circ(\nabla_{A\otimes V}\otimes W)\circ\nabla_{A\otimes V\otimes W}\circ(i_{A\otimes V}\otimes W)\\
&=\omega\circ p_{(A\times V)\otimes W},
\end{aligned}
$$

where the first equality follows because $\mu_{A\otimes V\otimes W}$ is normalized for $\nabla_{A\otimes V\otimes W}$ and by the associativity of μ_A, the second one follows by (74) and by the associativity of μ_A, the third one relies on (1) for $\mathbb{A}_V$ and the fourth one is a consequence of the properties of $\nabla_{A\otimes W}$. The fifth one follows by (9) and by the associativity of μ_A, the sixth one follows by (110), and in the seventh one we used the left linearity of $\nabla_{A\otimes V}$ and (1) for $\mathbb{A}_V$. In the eighth one we applied the left linearity of $\nabla_{A\otimes V}$ and (18) for ν_W. The ninth one follows by (66) and by (3), and the last one follows by definition.

The final assertion of this theorem follows by Theorem 4.1.

□

Example 4.3. In this example we will see that the equalities (108), (109) and (110) hold in the examples (3.1), (3.2) and (3.3) of the previous section.

For the Example (3.1) the identities (108), (109) and (110) hold because

$$\psi_T^S=\lambda_1,\ \sigma_T^S=\mu_T\circledcirc\eta_S,\ \tau_T^D=\lambda_2,\ \nu_D=\eta_S\circledcirc\eta_D,$$

and

$$\Delta_{T\circledcirc D}=id_{T\circledcirc D},\ \nabla_{S\circledcirc T\circledcirc D}=id_{S\circledcirc T\circledcirc D}.$$

In the case of the Example (3.2) we have that

$$\psi_T^S=\lambda_1,\ \sigma_T^S=\lambda_1\circ(\mu_T\circledcirc\eta_S),\ \tau_T^D=\lambda_2,\ \nu_D=\nabla_{S\circledcirc D}\circ(\eta_S\circledcirc\eta_D),$$

and $\Delta_{T\circledcirc D}=\nabla_{T\circledcirc D}$. Therefore, by the usual arguments, we obtain that (108), (109) and (110) hold because

$$
\begin{aligned}
&\nabla_{S\circledcirc T\circledcirc D}\circ(\mu_S\circledcirc T\circledcirc D)\circ(S\circledcirc\sigma_T^S\circledcirc D)\circ(\psi_S^T\circledcirc\tau_T^D)\circ(T\circledcirc\nu_D\circledcirc T)\\
&=(\lambda_1\circledcirc D)\circ(T\circledcirc\lambda_3)\circ((\nabla_{T\circledcirc D}\circ(\mu_T\circledcirc\eta_D))\circledcirc\eta_S)\\
&=\nabla_{S\circledcirc T\circledcirc D}\circ(\mu_S\circledcirc T\circledcirc D)\circ(S\circledcirc\psi_T^S\circledcirc D)\circ(\sigma_T^S\circledcirc\nu_D),
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{S\circledcirc T\circledcirc D}\circ(\sigma_T^S\circledcirc D)&=(S\circledcirc\mu_T\circledcirc D)\circ(\lambda_1\circledcirc\lambda_2)\circ(T\circledcirc\lambda_3\circledcirc T)\circ(\mu_T\circledcirc D\circledcirc(\lambda_1\circ(\eta_T\circledcirc\eta_S)))\\
&=(\mu_S\circledcirc T\circledcirc D)\circ(S\circledcirc\psi_T^S\circledcirc D)\circ(\sigma_T^S\circledcirc\psi_D^S)\circ(T\circledcirc\Delta_{T\circledcirc D}\circledcirc\eta_S),
\end{aligned}
$$

and

$$
\begin{aligned}
(\psi_T^S\circledcirc D)\circ(T\circledcirc\sigma_D^S)\circ(\Delta_{T\circledcirc D}\circledcirc D)&=(S\circledcirc\mu_T\circledcirc D)\circ(\lambda_1\circledcirc\lambda_2)\circ(T\circledcirc\lambda_3\circledcirc T)\circ(T\circledcirc\mu_D\circledcirc(\lambda_1\circ(\eta_T\circledcirc\eta_S)))\\
&=\nabla_{S\circledcirc T\circledcirc D}\circ(\psi_T^S\circledcirc D)\circ(T\circledcirc\sigma_D^S).
\end{aligned}
$$

Finally, in Example (3.3) we have that

$$\nu_W=\eta_A\otimes\eta_W,\ \Delta_{V\otimes W}=id_{V\otimes W},\ \nabla_{A\otimes V\otimes W}=id_{A\otimes V\otimes W},$$

and then (108), (109) and (110) follow easily.

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References

[1] A. Agore, G. Militaru, *Unified products and split extensions of Hopf algebras*, Contemp. Math. 585 (2013), 1-15.
[2] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, A.B., *Crossed products in weak contexts*, Appl. Categor. Struct. 3 (2010), 231-258.
[3] M. Beck, C. Wells, *Toposes, triples and theories*, Springer-Verlag, New York, (1985).
[4] J. Beck, *Distributive laws*, Springer LNM 80, 119-140 (1969).
[5] G. Böhm, *The weak theory of monads*, Adv. Math. 225 (2010), 1-32.
[6] G. Böhm, *On the iteration of weak wreath products*, Theor. Appl. Categ. 26 (2012), 30-59.
[7] G. Böhm, J. Gómez-Torrecillas, *Bilinear factorizations of algebras*, B. Belg. Math. Soc-Sim. 20 (2013), 1-24.
[8] G. Böhm, J. Gómez-Torrecillas, *On the double crossed product of weak Hopf algebras*, Contemp. Math. 585 (2013) 153-173.
[9] T. Brzeziński, *Crossed products by a coalgebra*, Commun. Algebra 25 (1997), 3551-3575.
[10] S. Caenepeel, E. De Groot, *Modules over weak entwining structures*, Contemp. Math. 267 (2000), 4701-4735.
[11] A. Cap, H. Schichl, J. Vanzura, *On twisted tensor products of algebras*, Commun. Algebra 23 (1995), 4701-4735.
[12] E. Cheng, *Iterated distributive laws*, Math. Proc. Cambridge Phil. Soc. 150 (2011), 459-487.
[13] L. Dăuş, F. Panaite, *A new way to iterate Brzeziński crossed products*, arXiv:1502.00031 (2015).
[14] J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, *Preunits and weak crossed products*, J. Pure Appl. Algebra 213 (2009), 2244-2261.
[15] J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, *Weak crossed biproducts and weak projections*, Sci. China Math. 55 (2012), 1443-11460
[16] J.M. Fernández Vilaboa, R. González Rodríguez, A.B. Rodríguez Raposo, *Partial and unified crossed products are weak crossed products*, Contemp. Math. 585 (2013) 261-274.
[17] P. Jara Martínez, J. López Peña, F. Panaite, F. van Oystaeyen, *On iterated tensor product of algebras*, Int. J. Math. 19 (2008), 1053-1101.
[18] C. Kassel, *Quantum Groups*, Springer-Verlag, 1995.
[19] S. Lack, R. Street, *The formal theory of monads II*, J. Pure Appl. Algebra 175 (2002), 243-265.
[20] J. López Peña, F. Panaite, F. van Oystaeyen, *General twisting of algebras*, Adv. Math. 212 (2007), 315-337.
[21] S. Maclane, *Categories for the Working Mathematicien*, Springer-Verlag, 1971.
[22] S. Majid, *More examples of bicrossproduct and double cross product Hopf algebras,* Isr. J. Math. 72 (1990), 133-148.
[23] M. Muniz S. Alves, E. Batista, M. Dokuchaev, A. Paques, *Twisted partial actions of Hopf algebras*, Isr. J. of Math. 197 (2013), 263-308.
[24] F. Panaite, *Iterated crossed products*, J. Algebra Appl. 13 (2014), 1450036 (14 pages)
[25] A.B. Rodríguez Raposo, *Crossed products for weak Hopf algebras*, Commun. Algebra 37 (2009), 2274-2289.
[26] R. Street, *The formal theory of monads*, J. Pure Appl. Algebra 2 (1972), 149-168.
[27] R. Street, *Weak distributive laws*, Theor. Appl. Categ. 22 (2009), 313-320.
[28] D. Tambara, *The endomorphism bialgebra of an algebra*, J. Fac. Univ. Tokyo Sect. IA 37 (1990), 425-456.