Quasigroupoids and weak Hopf quasigroups

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2. Weak Hopf quasigroups
3. Grouplike elements for weak Hopf quasigroups
4. Categorical equivalences
5. An isomorphism

Quasigroupoids

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2. Weak Hopf quasigroups
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Definition

A quasigroupoid A is an ordered pair of sets $A = (A_0, A_1)$ such that:

(1) There exist maps $s_A : A_1 \to A_0$, $t_A : A_1 \to A_0$, and $id_A : A_0 \to A_1$, called source, target and identity, respectively, satisfying

$$s_A(id_A(x)) = t_A(id_A(x)) = x, \quad \forall\, x \in A_0.$$

(2) There exist a map, called product of A,

$$\bullet : A_1 \,{}_{s_A}\!\times_{t_A} A_1 = \{(a, b) \in A_1 \times A_1 \,;\; s_A(a) = t_A(b)\} \to A_1,$$

defined by $\bullet(a, b) = a \bullet b$ and a map $\lambda_A : A_1 \to A_1$, called the inverse map, such that:

(2-1) For each $a \in \mathbf{A_1}$,

$$id_{\mathbf{A}}(t_{\mathbf{A}}(a)) \bullet a = a = a \bullet id_{\mathbf{A}}(s_{\mathbf{A}}(a)).$$

(2-2) For all $(a, b) \in \mathbf{A_1} \,{}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{A}}} \mathbf{A_1}$,

$$s_{\mathbf{A}}(a \bullet b) = s_{\mathbf{A}}(b), \quad t_{\mathbf{A}}(a \bullet b) = t_{\mathbf{A}}(a).$$

(2-3) For all $(a, b) \in \mathbf{A_1} \,{}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{A}}} \mathbf{A_1}$, $(\lambda_{\mathbf{A}}(a), a \bullet b)$ and $(a \bullet b, \lambda_{\mathbf{A}}(b))$ are in $\mathbf{A_1} \,{}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{A}}} \mathbf{A_1}$ and

$$\lambda_{\mathbf{A}}(a) \bullet (a \bullet b) = b, \quad (a \bullet b) \bullet \lambda_{\mathbf{A}}(b) = a.$$

The set A_0 wil be called the base of A. We will say that a quasigroupoid A is finite if its base is a finite set. Note that a finite quasigroupoid where $|A_0| = 1$ is an I.P. loop or a quasigroup in the sense of J. Klim and S. Majid.

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A quasigroupoid is an inverse semiloopoid satisfying the unities associativity assumption. These notions were introduced by J. Grabowski in

- **J. Grabowski.** An introduction to loopoids, Comment. Math. Univ. Carolin. 57 (2016), 515-526.

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We have that the following equalities:

$$
\begin{aligned}
s_A(\lambda_A(a)) &= t_A(a), \\
t_A(\lambda_A(a)) &= s_A(a), \\
\lambda_A(a) \bullet a &= id_A(s_A(a)), \\
a \bullet \lambda_A(a) &= id_A(t_A(a)), \\
\lambda_A(\lambda_A(a)) &= a, \\
\lambda_A(a \bullet b) &= \lambda_A(b) \bullet \lambda_A(a),
\end{aligned}
$$

hold for all $a \in A_1$ and $(a, b) \in A_1 \; {}_{s_A}\!\times_{t_A} A_1$.

Definition

Let A, A′ be quasigroupoids. A morphism $\Gamma : A \rightarrow A'$ between A and A′ is a pair of maps $\Gamma = (\Gamma_0, \Gamma_1)$, $\Gamma_0 : A_0 \rightarrow A'_0$, $\Gamma_1 : A_1 \rightarrow A'_1$, such that

(1) $\Gamma_0 \circ s_A = s_{A'} \circ \Gamma_1$,

(2) $\Gamma_0 \circ t_A = t_{A'} \circ \Gamma_1$,

(3) For all $x \in A_0$, $\Gamma_1(id_A(x)) = id_{A'}(\Gamma_0(x))$,

(4) For all $(a, b) \in A_1 \ {}_{s_A}\!\times_{t_A} A_1$, $\Gamma_1(a \bullet b) = \Gamma_1(a) \bullet' \Gamma_1(b)$.

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(4) For all $(a, b) \in A_1 \; {}_{s_A}\!\times_{t_A} A_1$, $\Gamma_1(a \bullet b) = \Gamma_1(a) \bullet' \Gamma_1(b)$.

The obvious composition of quasigroupoid morphisms is a quasigroupoid morphism. Then with QGPD we will denote the category whose objects are quasigroupoids and whose morphisms are morphisms of quasigroupoids. With $\overline{\text{QGPD}}$ we will denote the full subcategory of QGPD whose objects are finite quasigroupoids.

Example

Let A be a quasigroup (in the sense of J. Klim and S. Majid) with product $\cdot$ and let X be a set. Assume that there exists a map $\psi_X : A \times X \to X$ satisfying the following two conditions:

$$\psi_X(e_A, x) = x, \qquad \psi_X(a \cdot b, x) = \psi_X(a, \psi_X(b, x)),$$

for all $x \in X$ and $a, b \in A$.

In this case we will say that ψ_X is an action of A over X. The quasigroupoid $\mathrm{B} = (\mathrm{B}_0, \mathrm{B}_1)$ associated to the action ψ_X is defined by the sets $\mathrm{B}_0 = X$, $\mathrm{B}_1 = A \times X$ and maps

$$s_{\mathrm{B}} : \mathrm{B}_1 \to \mathrm{B}_0, \quad s_{\mathrm{B}}(a, x) = x,$$

$$t_{\mathrm{B}} : \mathrm{B}_1 \to \mathrm{B}_0, \quad t_{\mathrm{B}}(a, x) = \psi_X(a, x),$$

$$id_{\mathrm{B}} : \mathrm{B}_0 \to \mathrm{B}_1, \quad id_{\mathrm{B}}(x) = (e_A, x).$$

Then, $\mathrm{B}_1 \ {}_{s_{\mathrm{B}}}\!\times_{t_{\mathrm{B}}} \mathrm{B}_1 = \{((a, x), (b, y),) \in \mathrm{B}_1 \times \mathrm{B}_1 \ / \ \psi_X(b, y) = x\}$ and the product is defined by $(a, x) \star (b, y) = (a \cdot b, y)$. The inverse map $\lambda_{\mathrm{B}} : \mathrm{B}_1 \to \mathrm{B}_1$ is $\lambda_{\mathrm{B}}(a, x) = (a^{-1}, \psi_X(a, x))$.

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Then, $\mathrm{B}_1 \ {}_{s_{\mathrm{B}}}\!\times_{t_{\mathrm{B}}} \mathrm{B}_1 = \{((a, x), (b, y),) \in \mathrm{B}_1 \times \mathrm{B}_1 \ / \ \psi_X(b, y) = x\}$ and the product is defined by $(a, x) \star (b, y) = (a \cdot b, y)$. The inverse map $\lambda_{\mathrm{B}} : \mathrm{B}_1 \to \mathrm{B}_1$ is $\lambda_{\mathrm{B}}(a, x) = (a^{-1}, \psi_X(a, x))$.

As was proved in

- **Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R.** Quasigroupoids and weak Hopf quasigroups Journal of Algebra 568, 408-436 (2021)

examples of this kind can be obtained by working with Moufang loops of small order and the 4-dimensional Taft Hopf algebra.

Definition

Let A and H be quasigroupoids with products $\bullet$ and $\star$ and with the same base. A left action of H on A is a map $\varphi_A : \mathrm{H}_1 \ {}_{s_{\mathbf{H}}}\!\times_{t_{\mathbf{A}}} \mathrm{A}_1 \to \mathrm{A}_1$, satisfying:

(1) For all $(h, a) \in \mathrm{H}_1 \ {}_{s_{\mathbf{H}}}\!\times_{t_{\mathbf{A}}} \mathrm{A}_1$, $t_{\mathrm{A}}(\varphi_{\mathrm{A}}(h, a)) = t_{\mathrm{H}}(h)$.

(2) For all $(h, a) \in \mathrm{H}_1 \ {}_{s_{\mathbf{H}}}\!\times_{t_{\mathbf{A}}} \mathrm{A}_1$, $(g, h) \in \mathrm{H}_1 \ {}_{s_{\mathbf{H}}}\!\times_{t_{\mathbf{H}}} \mathrm{H}_1$, $\varphi_{\mathrm{A}}(g \star h, a) = \varphi_{\mathrm{A}}(g, \varphi_{\mathrm{A}}(h, a))$.

(3) For all $a \in \mathrm{A}_1$, $\varphi_{\mathrm{A}}(id_{\mathrm{H}}(t_{\mathrm{A}}(a)), a) = a$.

Definition

Let A and H be quasigroupoids with products $\bullet$ and $\star$ and with the same base. A left action of H on A is a map $\varphi_A : \mathsf{H}_1 \;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1 \to \mathsf{A}_1$, satisfying:

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(2) For all $(h, a) \in \mathsf{H}_1 \;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$, $(g, h) \in \mathsf{H}_1 \;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{H}}} \mathsf{H}_1$, $\varphi_{\mathsf{A}}(g \star h, a) = \varphi_{\mathsf{A}}(g, \varphi_{\mathsf{A}}(h, a))$.

(3) For all $a \in \mathsf{A}_1$, $\varphi_{\mathsf{A}}(id_{\mathsf{H}}(t_{\mathsf{A}}(a)), a) = a$.

Similarly, a right action of A on H is a map $\phi_{\mathsf{H}} : \mathsf{H}_1 \;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1 \to \mathsf{H}_1$, satisfying:

(1) For all $(h, a) \in \mathsf{H}_1 \;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$, $s_{\mathsf{H}}(\phi_{\mathsf{H}}(h, a)) = s_{\mathsf{A}}(a)$.

(2) For all $(h, a) \in \mathsf{H}_1 \;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$, $(a, b) \in \mathsf{A}_1 \;{}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$, $\phi_{\mathsf{H}}(h, a \bullet b) = \phi_{\mathsf{H}}(\phi_{\mathsf{H}}(h, a), b)$,

(3) For all $h \in \mathsf{H}_1$, $\phi_{\mathsf{H}}(h, id_{\mathsf{A}}(s_{\mathsf{H}}(h))) = h$.

Definition

A matched pair of quasigroupoids is a pair of quasigroupoids (A, H) with the same base together with a left action of H on A, $\varphi_{\mathsf{A}} : \mathsf{H}_1 \; {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1 \to \mathsf{A}_1$, and a right action of A on H, $\phi_{\mathsf{H}} : \mathsf{H}_1 \; {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1 \to \mathsf{H}_1$, satisfying the following properties:

(1) For all $(h, a) \in \mathsf{H}_1 \; {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$,

$$s_{\mathsf{A}}(\varphi_{\mathsf{A}}(h, a)) = t_{\mathsf{H}}(\phi_{\mathsf{H}}(h, a)).$$

(2) For all $(h, a) \in \mathsf{H}_1 \; {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$ and $(a, b) \in \mathsf{A}_1 \; {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$,

$$\varphi_{\mathsf{A}}(h, a \bullet b) = \varphi_{\mathsf{A}}(h, a) \bullet \varphi_{\mathsf{A}}(\phi_{\mathsf{H}}(h, a), b).$$

(3) For all $(h, a) \in \mathsf{H}_1 \; {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$ and $(g, h) \in \mathsf{H}_1 \; {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{H}}} \mathsf{H}_1$,

$$\phi_{\mathsf{H}}(g \star h, a) = \phi_{\mathsf{H}}(g, \varphi_{\mathsf{A}}(h, a)) \star \phi_{\mathsf{H}}(h, a).$$

Theorem

Let (A,H) be a matched pair of quasigroupoids. For all $a,b\in \mathrm{A}_1$, $g,h\in \mathrm{H}_1$ for which the operations are defined, the following identities hold:

$$
\begin{aligned}
\varphi_{\mathrm{A}}(h, id_{\mathrm{A}}(s_{\mathrm{H}}(h))) &= id_{\mathrm{A}}(t_{\mathrm{H}}(h)),\\
\phi_{\mathrm{H}}(id_{\mathrm{H}}(t_{\mathrm{A}}(a)), a) &= id_{\mathrm{H}}(s_{\mathrm{A}}(a)),\\
\lambda_{\mathrm{A}}(\varphi_{\mathrm{A}}(h,a)) &= \varphi_{\mathrm{A}}(\phi_{\mathrm{H}}(h,a),\lambda_{\mathrm{A}}(a)),\\
\lambda_{\mathrm{H}}(\phi_{\mathrm{H}}(h,a)) &= \phi_{\mathrm{H}}(\lambda_{\mathrm{H}}(h),\varphi_{\mathrm{A}}(h,a))\\
(b\bullet\varphi_{\mathrm{A}}(h,a))\bullet\varphi_{\mathrm{A}}(\phi_{\mathrm{H}}(h,a),\lambda_{\mathrm{A}}(a)) &= b,\\
\phi_{\mathrm{H}}(\lambda_{\mathrm{H}}(h),\varphi_{\mathrm{A}}(h,a))\star(\phi_{\mathrm{H}}(h,a)\star g) &= g,\\
\varphi_{\mathrm{A}}(\lambda_{\mathrm{H}}(\phi_{\mathrm{H}}(h,a)),\lambda_{\mathrm{A}}(\varphi_{\mathrm{A}}(h,a))) &= \lambda_{\mathrm{A}}(a),\\
\phi_{\mathrm{H}}(\lambda_{\mathrm{H}}(\phi_{\mathrm{H}}(h,a)),\lambda_{\mathrm{A}}(\varphi_{\mathrm{A}}(h,a))) &= \lambda_{\mathrm{H}}(h),\\
\lambda_{\mathrm{A}}(a)\bullet\varphi_{\mathrm{A}}(\lambda_{\mathrm{H}}(h),b) &= \varphi_{\mathrm{A}}(\lambda_{\mathrm{H}}(\phi_{\mathrm{H}}(h,a)),\lambda_{\mathrm{A}}(\varphi_{\mathrm{A}}(h,a))\bullet b),\\
\phi_{\mathrm{H}}(g,\lambda_{\mathrm{A}}(a))\star\lambda_{\mathrm{H}}(h) &= \phi_{\mathrm{H}}(g\star\lambda_{\mathrm{H}}(\phi_{\mathrm{H}}(h,a)),\lambda_{\mathrm{A}}(\varphi_{\mathrm{A}}(h,a))).
\end{aligned}
$$

Theorem

Let (A, H) be a matched pair of quasigroupoids. The pair

$$\mathrm{A} \bowtie \mathrm{H} = ((\mathrm{A} \bowtie \mathrm{H})_0, (\mathrm{A} \bowtie \mathrm{H})_1),$$

where $(\mathrm{A} \bowtie \mathrm{H})_0 = \mathrm{A}_0$, $(\mathrm{A} \bowtie \mathrm{H})_1 = \mathrm{A}_1 \ {}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{H}}} \mathrm{H}_1$, is a quasigroupoid with source morphism $s_{\mathrm{A} \bowtie \mathrm{H}}(a, h) = s_{\mathrm{H}}(h)$, target morphism $t_{\mathrm{A} \bowtie \mathrm{H}}(a, h) = t_{\mathrm{A}}(a)$, identity map $id_{\mathrm{A} \bowtie \mathrm{H}}(x) = (id_{\mathrm{A}}(x), id_{\mathrm{H}}(x))$, product

$$(a, g).{}_{\Psi}(b, h) = (a \bullet \Psi_1(g, b), \Psi_2(g, b) \star h),$$

where $\Psi : \mathrm{H}_1 \ {}_{s_{\mathbf{H}}}\!\times_{t_{\mathbf{A}}} \mathrm{A}_1 \rightarrow \mathrm{A}_1 \ {}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{H}}} \mathrm{H}_1$ is the map with components $\Psi_1(g, b) = \varphi_{\mathrm{A}}(g, b)$ and $\Psi_2(g, b) = \phi_{\mathrm{H}}(g, b)$, and inverse map

$$\lambda_{\mathrm{A} \bowtie \mathrm{H}}(a, h) = \Psi(\lambda_{\mathrm{H}}(h), \lambda_{\mathrm{A}}(a)).$$

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$$(a, g) ._{\Psi} (b, h) = (a \bullet \Psi_1(g, b), \Psi_2(g, b) \star h),$$

where $\Psi : \mathrm{H}_1 \;{}_{s_{\mathbf{H}}}\!\times_{t_{\mathbf{A}}} \mathrm{A}_1 \to \mathrm{A}_1 \;{}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{H}}} \mathrm{H}_1$ is the map with components $\Psi_1(g, b) = \varphi_{\mathrm{A}}(g, b)$ and $\Psi_2(g, b) = \phi_{\mathrm{H}}(g, b)$, and inverse map

$$\lambda_{\mathrm{A} \bowtie \mathrm{H}}(a, h) = \Psi(\lambda_{\mathrm{H}}(h), \lambda_{\mathrm{A}}(a)).$$

Definition

Let (A, H) be a matched pair of quasigroupoids. The quasigroupoid $\mathrm{A} \bowtie \mathrm{H}$ will be called the double crossed product or the diagonal quasigroupoid of A and H.

Theorem

Let (A, H) be a matched pair of quasigroupoids. Then, A and H are subquasigroupoids of $\mathsf{A} \bowtie \mathsf{H}$ where

$$i^{\mathsf{A}} : \mathsf{A} \to \mathsf{A} \bowtie \mathsf{H}$$

is defined by $i^{\mathsf{A}} = (i_0^{\mathsf{A}} : \mathsf{A}_0 \to (\mathsf{A} \bowtie \mathsf{H})_0, i_1^{\mathsf{A}} : \mathsf{A}_1 \to (\mathsf{A} \bowtie \mathsf{H})_1)$ with

$$i_0^{\mathsf{A}} = id_{\mathsf{A_0}}, \quad i_1^{\mathsf{A}}(a) = (a, id_{\mathsf{H}}(s_{\mathsf{A}}(a))),$$

and $i^{\mathsf{H}} : \mathsf{H} \to \mathsf{A} \bowtie \mathsf{H}$ is defined by $i^{\mathsf{H}} = (i_0^{\mathsf{H}} : \mathsf{A}_0 \to (\mathsf{A} \bowtie \mathsf{H})_0, i_1^{\mathsf{H}} : \mathsf{H}_1 \to (\mathsf{A} \bowtie \mathsf{H})_1)$ where

$$i_0^{\mathsf{H}} = id_{\mathsf{A_0}}, \quad i_1^{\mathsf{H}}(g) = (id_{\mathsf{A}}(t_{\mathsf{H}}(g)), g),$$

Definition

Let A, H and B be quasigroupoids with the same base. Assume that A and H are subquasigroupoids of B with associated monomorphisms $i^{\mathsf{A}} : \mathsf{A} \to \mathsf{B}$ and $i^{\mathsf{H}} : \mathsf{H} \to \mathsf{B}$ satisfying that $i_0^{\mathsf{A}} = i_0^{\mathsf{H}} = id_{\mathsf{A_0}}$. Let $\diamond$ be the product defined in B. Then we will say that $[\mathsf{A}, \mathsf{H}]$ is an exact factorization of B, if in the case that the products involved are defined, we have that

(i) $i^{\mathsf{H}}(g) \diamond (i^{\mathsf{A}}(a) \diamond i^{\mathsf{A}}(b)) = (i^{\mathsf{H}}(g) \diamond i^{\mathsf{A}}(a)) \diamond i^{\mathsf{A}}(b)$,

(ii) $i^{\mathsf{H}}(g) \diamond (i^{\mathsf{H}}(h) \diamond i^{\mathsf{A}}(a)) = (i^{\mathsf{H}}(g) \diamond i^{\mathsf{H}}(h)) \diamond i^{\mathsf{A}}(a)$,

(iii) $i^{\mathsf{H}}(h) \diamond (i^{\mathsf{A}}(a) \diamond i^{\mathsf{H}}(f)) = (i^{\mathsf{H}}(h) \diamond i^{\mathsf{A}}(a)) \diamond i^{\mathsf{H}}(f)$,

(iv) $i^{\mathsf{A}}(c) \diamond (i^{\mathsf{H}}(h) \diamond i^{\mathsf{A}}(a)) = (i^{\mathsf{A}}(c) \diamond i^{\mathsf{H}}(h)) \diamond i^{\mathsf{A}}(a)$,

(v) $i^{\mathsf{A}}(a) \diamond (i^{\mathsf{A}}(b) \diamond i^{\mathsf{H}}(g)) = (i^{\mathsf{A}}(a) \diamond i^{\mathsf{A}}(b)) \diamond i^{\mathsf{H}}(g)$,

(vi) $i^{\mathsf{A}}(a) \diamond (i^{\mathsf{H}}(g) \diamond i^{\mathsf{H}}(h)) = (i^{\mathsf{A}}(a) \diamond i^{\mathsf{H}}(h)) \diamond i^{\mathsf{H}}(g)$

hold and

(vii) The map $\theta_{\mathsf{B}} = \diamond \circ (i^{\mathsf{A}} \times i^{\mathsf{H}}) : \mathsf{A}_1 \, {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{H}}} \mathsf{H}_1 \to \mathsf{B}_1$ is a bijection.

Theorem

Let $[A, H]$ be an exact factorization of a quasigropupoid B. Then, there exists a matched pair of quasigroupoids (A, H) and an isomorphism of quasigroupoids between $A \bowtie H$ and B.

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Example

Let A be a quasigroup with product $\cdot$ and let X be a set. Assume that there exists an action of A over X denoted by $\psi_X : A \times X \to X$. Let $B = (B_0, B_1)$ be the quasigroupoid associated to the action ψ_X. Let X^d be the discrete groupoid. Then, (X^d, B) is a matched pair of quasigroupoids where $\varphi_{X^d}((a, x), x) = \psi_X(a, x)$ and $\phi_B((a, x), x) = (a, x)$. In this case the diagonal quasigroupoid $X^d \bowtie B$, is defined by

$$(X^d \bowtie B)_0 = X, \quad (X^d \bowtie B)_1 = X_1^d \; {}_{s_{X^d}}\!\times_{t_B} B_1,$$

$$s_{X^d \bowtie B}(y, (a, x)) = x, \quad t_{X^d \bowtie B}(y, (a, x)) = y, \quad id_{X^d \bowtie B}(x) = (x, (e_A, x)),$$

$$(y, (a, x))._{\Psi}(x, (b, t)) = (y, (a \cdot b, t))$$

and

$$\lambda_{X^d \bowtie B}(y, (a, x)) = (\psi_X(a^{-1}, y), (a^{-1}, y)).$$

Weak Hopf quasigroups

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3. Grouplike elements for weak Hopf quasigroups
4. Categorical equivalences
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Definition

A weak Hopf quasigroup in $\mathbb{K}$-Vect is a $\mathbb{K}$-vector space H such that it is a unital magma with product $\mu_H(h\otimes g) = hg$ and unit 1 and a coalgebra with coproduct Δ_H and counit ε_H, satisfying the following conditions for all $h, k, l \in H$:

(1) $(hk)_{(1)} \otimes (hk)_{(2)} = h_{(1)}k_{(1)} \otimes h_{(2)}k_{(2)}$.

(2) $\varepsilon_H((hk)l) = \varepsilon_H(h(kl)) = \varepsilon_H(hk_{(1)})\varepsilon_H(k_{(2)}l) = \varepsilon_H(hk_{(2)})\varepsilon_H(k_{(1)}l)$.

(3) $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')}$.

(4) There exists a linear map $\lambda_H : H \to H$ (called the antipode of H) such that, if $\Pi_H^L : H \to H$ is the $\mathbb{K}$-linear map defined by $\Pi_H^L = id_H * \lambda_H$ (target morphism) and $\Pi_H^R : H \to H$ is the $\mathbb{K}$-linear map defined by $\Pi_H^R = \lambda_H * id_H$ (source morphism),

- (4-1) $\Pi_H^L(h) = \varepsilon_H(1_{(1)}h)1_{(2)}$.
- (4-2) $\Pi_H^R(h) = \varepsilon_H(h1_{(2)})1_{(1)}$.
- (4-3) $\lambda_H = \lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H$.
- (4-4) $\lambda_H(h_{(1)})(h_{(2)}k) = \Pi_H^R(h)k$.
- (4-5) $h_{(1)}(\lambda_H(h_{(2)})k) = \Pi_H^L(h)k$.
- (4-6) $(hk_{(1)})\lambda_H(k_{(2)}) = h\Pi_H^L(k)$.
- (4-7) $(h\lambda_H(k_{(1)}))k_{(2)} = h\Pi_H^R(k)$.

Definition

A weak Hopf quasigroup in $\mathbb{K}$-Vect is a $\mathbb{K}$-vector space H such that it is a unital magma with product $\mu_H(h\otimes g) = hg$ and unit 1 and a coalgebra with coproduct Δ_H and counit ε_H, satisfying the following conditions for all $h, k, l \in H$:

(1) $(hk)_{(1)} \otimes (hk)_{(2)} = h_{(1)}k_{(1)} \otimes h_{(2)}k_{(2)}$.

(2) $\varepsilon_H((hk)l) = \varepsilon_H(h(kl)) = \varepsilon_H(hk_{(1)})\varepsilon_H(k_{(2)}l) = \varepsilon_H(hk_{(2)})\varepsilon_H(k_{(1)}l)$.

(3) $1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')}1_{(2)} \otimes 1_{(2')}$.

(4) There exists a linear map $\lambda_H : H \to H$ (called the antipode of H) such that, if $\Pi_H^L : H \to H$ is the $\mathbb{K}$-linear map defined by $\Pi_H^L = id_H * \lambda_H$ (target morphism) and $\Pi_H^R : H \to H$ is the $\mathbb{K}$-linear map defined by $\Pi_H^R = \lambda_H * id_H$ (source morphism),

- (4-1) $\Pi_H^L(h) = \varepsilon_H(1_{(1)}h)1_{(2)}$.
- (4-2) $\Pi_H^R(h) = \varepsilon_H(h1_{(2)})1_{(1)}$.
- (4-3) $\lambda_H = \lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H$.
- (4-4) $\lambda_H(h_{(1)})(h_{(2)}k) = \Pi_H^R(h)k$.
- (4-5) $h_{(1)}(\lambda_H(h_{(2)})k) = \Pi_H^L(h)k$.
- (4-6) $(hk_{(1)})\lambda_H(k_{(2)}) = h\Pi_H^L(k)$.
- (4-7) $(h\lambda_H(k_{(1)}))k_{(2)} = h\Pi_H^R(k)$.

- J. N. Alonso Álvarez, J. M. Fernández Vilaboa, R. González Rodríguez, Weak Hopf quasigroups, Asian J. of Math. 20 (2016), 665-694.

Example

Let $\mathbb{K}$ be a field and let $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$ be a finite quasigroupoid. The quasigroupoid magma $\mathbb{K}[\mathsf{A}]$ defined by

$$\mathbb{K}[\mathsf{A}] = \bigoplus_{a \in \mathsf{A}_1} \mathbb{K}a$$

is a cocommutative weak Hopf quasigroup with unit $1 = \sum_{x \in \mathsf{A}_0} id_{\mathsf{A}}(x)$, product

$$\mu_{\mathbb{K}[\mathsf{A}]}(a \otimes b) = \begin{cases} a \bullet b, & \text{if } (a, b) \in \mathsf{A}_1 \; {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1, \\ 0, & \text{if } (a, b) \notin \mathsf{A}_1 \; {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1, \end{cases}$$

counit $\varepsilon_{\mathbb{K}[\mathsf{A}]}(a) = 1_{\mathbb{K}}$, coproduct $\Delta_{\mathbb{K}[\mathsf{A}]}(a) = a \otimes a$ and antipode $\lambda_{\mathbb{K}[\mathsf{A}]}(a) = \lambda_{\mathsf{A}}(a)$ on the basis elements.

If H is a weak Hopf quasigroup, the target and source maps are idempotent.

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The antipode of a weak Hopf quasigroup H is unique, $\lambda_H \circ \eta_H = \eta_H$, $\varepsilon_H \circ \lambda_H = \varepsilon_H$ and is antimultiplicative and anticomultiplicative, i.e.,

$$\lambda_H(hg) = \lambda_H(g)\lambda_H(h),$$
$$\lambda_H(h)_{(1)} \otimes \lambda_H(h)_{(2)} = \lambda_H(h_{(2)}) \otimes \lambda_H(h_{(1)})$$

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If H_L is the subspace defined by the image of the target morphism and $h \in H_L$, the following identities hold:

$$(hk)l = h(kl), \quad k(hl) = (kh)l, \quad k(lh) = (kl)h,$$

for all $k, l \in H$.

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for all $k, l \in H$.

As a consequence, the unital magma H_L is an algebra in $\mathbb{K}$-Vect, where the unit is $1_{H_L} = \Pi_H^L(1)$ and $\mu_{H_L} = \Pi_H^L \circ \mu_H$.

For the image of the source morphism, denoted by H_R, we have similar properties.

Theorem

Let H be a weak Hopf quasigroup. The algebras H_L and H_R are separable, finite dimensional and semisimple.

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As was proved in

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weak Hopf quasigroups and weak Hopf algebras satisfy similar properties for the morphisms target, source and $\overline{\Pi}_H^L, \overline{\Pi}_H^R : H \to H$, where

$$\overline{\Pi}_H^L(h) = \varepsilon_H(1_{(2)}h)1_{(1)}, \quad \overline{\Pi}_H^R(h) = \varepsilon_H(h1_{(1)})1_{(2)}.$$

Definition

Let H, H' be weak Hopf quasigroups. We will say that a $\mathbb{K}$-linear map $f : H \to H'$ is a morphism of weak Hopf quasigroups if it is a coalgebra morphism such that

$$\Pi^R_{H'} \circ f = f \circ \Pi^R_H,$$

$$\overline{\Pi}^L_{H'} \circ f = f \circ \overline{\Pi}^L_H,$$

$$\Pi^R_{H'} \circ \Pi^L_{H'} \circ f = f \circ \Pi^R_H \circ \Pi^L_H,$$

$$f \circ \mu_H = \mu_{H'} \circ (f \otimes f) \circ \nabla_H,$$

hold, where $\nabla_H : H \otimes H \to H \otimes H$ is the idempotent $\mathbb{K}$-linear map defined by

$$\nabla_H(h \otimes k) = h_{(1)} \otimes \Pi^R_H(h_{(2)})k.$$

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- The previous definition is similar to the one introduced in
  - **G. Böhm, J. Gómez-Torrecillas, E. López Centella,** On the category of weak bialgebras, J. Algebra 399 (2014), 801-844.

  for weak Hopf algebras.

Theoorem

Let $\mathbb{K}$ be a field and let $\Gamma = (\Gamma_0, \Gamma_1) : A \to A'$ be a morphism of finite quasigroupoids. Then the linear extension $\mathbb{K}[\Gamma_1] : \mathbb{K}[A] \to \mathbb{K}[A']$, $\mathbb{K}[\Gamma_1](a) = \Gamma_1(a)$, is a morphism of weak Hopf quasigroups that we will denote by $\mathbb{K}[\Gamma]$.

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Theorem

There exists a functor, called the WHQ-functor,

$$F : \overline{QGPD} \to WHQ$$

defined on objects by $F(A) = \mathbb{K}[A]$ and on morphisms by $F(\Gamma) = \mathbb{K}[\Gamma]$.

Grouplike elements for weak Hopf quasigroups

1. Quasigroupoids
2. Weak Hopf quasigroups
3. Grouplike elements for weak Hopf quasigroups
4. Categorical equivalences
5. An isomorphism

Let H be a weak Hopf quasigroup and $\mathrm{G}(H)$ its set of group-like elements as a coalgebra. If $\mathrm{g} \in H$ is a grouplike element of H, then so are $\Pi_H^L(\mathrm{g})$, $\Pi_H^R(\mathrm{g})$ and $\lambda_H(\mathrm{g})$.

Let H be a weak Hopf quasigroup and $\mathrm{G}(H)$ its set of group-like elements as a coalgebra. If $\mathrm{g} \in H$ is a grouplike element of H, then so are $\Pi_H^L(\mathrm{g})$, $\Pi_H^R(\mathrm{g})$ and $\lambda_H(\mathrm{g})$.

The group-like elements of H are linearly independent and the subspace H_R is a finite dimensional algebra. Therefore, the cardinality of $\mathrm{G}(H) \cap H_R$ is finite.

Theorem

Let H be a weak Hopf quasigroup. The ordered pair of sets $\mathcal{T}(H) = (\mathcal{T}(H)_0, \mathcal{T}(H)_1)$, where

$$\mathcal{T}(H)_0 = \mathrm{G}(H) \cap H_R, \quad \mathcal{T}(H)_1 = \mathrm{G}(H),$$

is a finite quasigroupoid where the source, target and identity morphisms

$$\mathfrak{s} : \mathcal{T}(H)_1 \to \mathcal{T}(H)_0, \quad \mathfrak{t} : \mathcal{T}(H)_1 \to \mathcal{T}(H)_0, \quad \mathfrak{id} : \mathcal{T}(H)_0 \to \mathcal{T}(H)_1,$$

are defined by

$$\mathfrak{s}(\mathrm{g}) = \Pi_H^R(\mathrm{g}), \quad \mathfrak{t}(\mathrm{g}) = \overline{\Pi}_H^L(\mathrm{g}), \quad \mathfrak{id}(\mathrm{r}) = \mathrm{r},$$

the product

$$\star : \mathcal{T}(H)_1 \;{}_{\mathfrak{s}}\times_{\mathfrak{t}}\; \mathcal{T}(H)_1 \to \mathcal{T}(H)_1$$

is defined by $\mathrm{h} \star \mathrm{g} = \mathrm{hg}$, and the inverse map $\lambda : \mathcal{T}(H)_1 \to \mathcal{T}(H)_1$ is $\lambda(\mathrm{g}) = \lambda_H(\mathrm{g})$.

Theorem

Let $f : H \to H'$ be a morphism of weak Hopf quasigroups. The pair $\mathcal{T}(f) = (\mathcal{T}(f)_0, \mathcal{T}(f)_1)$, where

$$\mathcal{T}(f)_0 : \mathcal{T}(H)_0 \to \mathcal{T}(H')_0, \quad \mathcal{T}(f)_1 : \mathcal{T}(H)_1 \to \mathcal{T}(H')_1$$

are the maps defined by $\mathcal{T}(f)_0(\mathrm{r}) = f(\mathrm{r})$ and $\mathcal{T}(f)_1(\mathrm{g}) = f(\mathrm{g})$, is a morphism of quasigroupoids between $\mathcal{T}(H)$ and $\mathcal{T}(H')$.

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Theorem

There exists a functor, called the QGPD-functor,

$$\mathrm{L} : \mathrm{WHQ} \to \overline{\mathrm{QGPD}}$$

defined on objects by $\mathrm{L}(H) = \mathcal{T}(H)$ and on morphisms by $\mathrm{L}(f) = \mathcal{T}(f)$.

Theorem

The WHQ-functor is left adjoint of the QGPD-functor.

Categorical equivalences

1. Quasigroupoids
2. Weak Hopf quasigroups
3. Grouplike elements for weak Hopf quasigroups
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Theorem

The functors F and L induce an equivalence between the category $\overline{\mathrm{QGPD}}$ and the full subcategory of WHQ of all pointed cosemisimple weak Hopf quasigroups over a given field $\mathbb{K}$.

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The functors F and L induce an equivalence between the category $\overline{\text{QGPD}}$ and the full subcategory of WHQ of all cocommutative cosemisimple weak Hopf quasigroups over an algebraically closed field $\mathbb{K}$.

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- In the associative setting (replace qusigroupoids by groupoids), the previous results are the ones proved in
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Theorem

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The category of quasigroups is equivalent to the one of cocommutative cosemisimple Hopf quasigroups over an algebraically closed field $\mathbb{K}$.

An isomorphism

1. Quasigroupoids
2. Weak Hopf quasigroups
3. Grouplike elements for weak Hopf quasigroups
4. Categorical equivalences
5. An isomorphism

Let (A, H) be a matched pair of finite quasigroupoids. Let φ_{A} be the left action of H on A and let ϕ_{H} be a right action of A on H. Then, we can define the following morphisms in $\mathbb{K}$-Vect by:

$$\varphi_{\mathbb{K}[\mathsf{A}]} : \mathbb{K}[\mathsf{H}] \otimes \mathbb{K}[\mathsf{A}] \to \mathbb{K}[\mathsf{A}], \quad \varphi_{\mathbb{K}[\mathsf{A}]}(h \otimes a) = \begin{cases} \varphi_{\mathsf{A}}(h, a) & \text{if } (h, a) \in \mathsf{H}_1 \ {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{\mathbb{K}[\mathsf{H}]} : \mathbb{K}[\mathsf{H}] \otimes \mathbb{K}[\mathsf{A}] \to \mathbb{K}[\mathsf{A}], \quad \phi_{\mathbb{K}[\mathsf{H}]}(h \otimes a) = \begin{cases} \phi_{\mathsf{H}}(h, a) & \text{if } (h, a) \in \mathsf{H}_1 \ {}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1 \\ 0 & \text{otherwise.} \end{cases}$$

The pair $(\mathbb{K}[\mathsf{A}], \varphi_{\mathbb{K}[\mathsf{A}]})$ is an example of left module over the unitary magma $\mathbb{K}[\mathsf{H}]$. Similarly, $(\mathbb{K}[\mathsf{H}], \phi_{\mathbb{K}[\mathsf{H}]})$ is an example of right module over the unitary magma $\mathbb{K}[\mathsf{A}]$.

Let $\Phi : \mathbb{K}[H] \otimes \mathbb{K}[A] \rightarrow \mathbb{K}[A] \otimes \mathbb{K}[H]$, $\nabla_{\Phi} : \mathbb{K}[A] \otimes \mathbb{K}[H] \rightarrow \mathbb{K}[A] \otimes \mathbb{K}[H]$ be the morphisms in $\mathbb{K}$-Vect defined by

$$\Phi = (\varphi_{\mathbb{K}[A]} \otimes \phi_{\mathbb{K}[H]}) \circ (id_{\mathbb{K}[A]} \otimes c_{\mathbb{K}[H],\mathbb{K}[A]} \otimes id_{\mathbb{K}[H]}) \circ (\delta_{\mathbb{K}[H]} \otimes \delta_{\mathbb{K}[A]})$$

and

$$\nabla_{\Phi} = (\mu_{\mathbb{K}[A]} \otimes id_{\mathbb{K}[H]}) \circ (id_{\mathbb{K}[A]} \otimes (\Phi \circ (id_{\mathbb{K}[H]} \otimes 1_{\mathbb{K}[A]}))).$$

Then,

$$\Phi(h\otimes a)=\begin{cases}\varphi_{\mathsf{A}}(h,a)\otimes\phi_{\mathsf{H}}(h,a) & \text{if } (h,a)\in \mathsf{H}_1\;{}_{s_{\mathsf{H}}}\!\times_{t_{\mathsf{A}}}\mathsf{A}_1\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\nabla_{\Phi}(a\otimes h)=\begin{cases}a\otimes h & \text{if } (a,h)\in \mathsf{A}_1\;{}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{H}}}\mathsf{H}_1\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, ∇_{Φ} is an idempotent morphism with image

$$Im(\nabla_{\Phi}) = \langle\{a \otimes h \;\; / \;\; (a, h) \in \mathsf{A}_{\mathbf{1}} \; {}_{s_{\mathbf{A}}}\!\times_{t_{\mathbf{H}}} \mathsf{H}_{\mathbf{1}}\}\rangle.$$

Therefore, ∇_{Φ} is an idempotent morphism with image

$$Im(\nabla_{\Phi}) = \langle \{a \otimes h \ / \ (a,h) \in A_1 \ {}_{s_A}\!\times_{t_H} H_1\} \rangle.$$

If we denote by $\mathbb{K}[A] \bowtie \mathbb{K}[H]$ the image of ∇_{Φ}, in this $\mathbb{K}$-vector space we can define a product by

$$\mu_{\mathbb{K}[A]\bowtie\mathbb{K}[H]} = (\mu_{\mathbb{K}[A]} \otimes \mu_{\mathbb{K}[H]}) \circ (id_{\mathbb{K}[A]} \otimes \Phi \otimes id_{\mathbb{K}[H]}).$$

Then,

$$\mu_{\mathbb{K}[A]\bowtie\mathbb{K}[H]}(a \otimes h \otimes b \otimes g) = \begin{cases} a \bullet \varphi_A(h,b) \otimes \phi_A(h,b) \star g & \text{if } \ (h,b) \in H_1 \ {}_{s_H}\!\times_{t_A} A_1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Let (A, H) be a matched pair of finite quasigroupoids. Then

$$\mathbb{K}[\mathsf{A}] \bowtie \mathbb{K}[\mathsf{H}]$$

is a cocommutative weak Hopf quasigroup with unit

$$1_{\mathbb{K}[\mathsf{A}]\bowtie\mathbb{K}[\mathsf{H}]} = \sum_{x\in \mathsf{A}_{\mathbf{0}}} id_{\mathsf{A}}(x) \otimes id_{\mathsf{H}}(x),$$

product $\mu_{\mathbb{K}[\mathsf{A}]\bowtie\mathbb{K}[\mathsf{H}]}$ and where for all not nulll element $a \otimes h \in \mathbb{K}[\mathsf{A}] \bowtie \mathbb{K}[\mathsf{H}]$, the counit is defined by $\varepsilon_{\mathbb{K}[\mathsf{A}]\bowtie\mathbb{K}[\mathsf{H}]}(a \otimes h) = 1$, the coproduct by $\delta_{\mathbb{K}[\mathsf{A}]\bowtie\mathbb{K}[\mathsf{H}]}(a \otimes h) = a \otimes h \otimes a \otimes h$ and the antipode by

$$\lambda_{\mathbb{K}[\mathsf{A}]\bowtie\mathbb{K}[\mathsf{H}]}(a \otimes h) = \varphi_{\mathsf{A}}(\lambda_{\mathsf{H}}(h), \lambda_{\mathsf{A}}(a)) \otimes \phi_{\mathsf{H}}(\lambda_{\mathsf{H}}(h), \lambda_{\mathsf{A}}(a)).$$

Theorem

Let (A, H) be a matched pair of finite quasigroupoids. The cocommutative weak Hopf quasigroups $\mathbb{K}[A \bowtie H]$ and $\mathbb{K}[A] \bowtie \mathbb{K}[H]$ are isomorphic in WHQ.

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Corollary

Let (A, H) be a matched pair of quasigroups. The cocommutative Hopf quasigroups $\mathbb{K}[A \bowtie H]$ and $\mathbb{K}[A] \bowtie \mathbb{K}[H]$ are isomorphic in the category of Hopf quasigroups.

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Let (A, H) be a matched pair of finite quasigroupoids. The cocommutative weak Hopf quasigroups $\mathbb{K}[\mathrm{A} \bowtie \mathrm{H}]$ and $\mathbb{K}[\mathrm{A}] \bowtie \mathbb{K}[\mathrm{H}]$ are isomorphic in WHQ.

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Thank you