Weak Hopf quasigroups, fusion morphisms and finite quasigroupoids

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Outline

Preliminaries

- From now on C denotes a braided monoidal category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the braiding.

  Recall that a monoidal category is a category C equipped with a tensor product functor $\otimes : C \times C \to C$, a unit object K of C and a family of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),$$

$$r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

  in C (called associativity, right unit and left unit constraints, respectively) satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

  where id_X denotes the identity morphism for each object X in C.

  On the other hand, C is braided if for any couple (M, N) of objects in the category there exists a natural isomorphism

$$c_{M,N} : M \otimes N \to N \otimes M$$

  satisfying the Hexagon Axiom

$$a_{N,P,M} \circ c_{M,N\otimes P} \circ a_{M,N,P} = (id_N \otimes c_{M,P}) \circ a_{N,M,P} \circ (c_{M,N} \otimes id_P),$$

$$a^{-1}_{P,M,N} \circ c_{M\otimes N,P} \circ a^{-1}_{M,N,P} = (c_{M,P} \otimes id_N) \circ a^{-1}_{M,P,N} \circ (id_M \otimes c_{N,P}),$$

  for all M, N and P in C.

- Taking into account that every non-strict monoidal category is monoidal equivalent to a strict one (the constraints are identities), we can assume without loss of generality that the category is strict and, as a consequence, the results contained in this talk remain valid for every non-strict braided monoidal category.
- We also assume that every idempotent morphism $q : Y \rightarrow Y$ in C splits (C is Cauchy complete), i.e. there exist an object Z (called the image of q) and morphisms $i : Z \rightarrow Y$ and $p : Y \rightarrow Z$ such that $q = i \circ p$ and $p \circ i = id_Z$.

  The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with equalizers or coequalizers.

- For simplicity of notation, given three objects V, U, M in C and a morphism $f : V \rightarrow U$, we write

$$M \otimes f \text{ for } id_M \otimes f \text{ and } f \otimes M \text{ for } f \otimes id_M.$$

- The triple (A, η_A, μ_A) is a unital magma, if $\eta_A : K \to A$ (unit) and $\mu_A : A \otimes A \to A$ (product) are morphisms in C such that

$$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A).$$

If (A, η_A, μ_A) is a unital magma and

$$\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$$

we will say that A is an algebra.

Let (A, η_A, μ_A) and (B, η_B, μ_B) be unital magmas (algebras). A morphism

$$f : A \to B$$

in C is a morphism of unital magmas if

$$\eta_B = f \circ \eta_A, \quad f \circ \mu_A = \mu_B \circ (f \otimes f).$$

- The triple $(C, \varepsilon_C, \delta_C)$ is a counital comagma with comultiplication δ_C and counit ε_C if

$$(\varepsilon_C \otimes C) \circ \delta_C = id_C = (C \otimes \varepsilon_C) \circ \delta_C$$

If moreover, the following identity holds

$$(\delta_C \otimes C) \circ \delta_C = (C \otimes \delta_C) \circ \delta_C$$

we will say that $(C, \varepsilon_C, \delta_C)$ is a coalgebra.

Let $(C, \varepsilon_C, \delta_C)$ and $(D, \varepsilon_D, \delta_D)$ be counital comagmas (coalgebras). A morphism

$$f : C \to D$$

in C is a morphism of counital comagmas if

$$\varepsilon_C = \varepsilon_D \circ f, \quad \delta_D \circ f = (f \otimes f) \circ \delta_C.$$

- If $f, g : C \to A$ are morphisms between a counital comagma C and a unital magma A,

$$f * g$$

denotes the convolution product defined by $f * g = \mu_A \circ (f \otimes g) \circ \delta_C$.

- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez**: Weak Hopf quasigroups, Asian Journal of Mathematics 20, N. 4, 665-694 (2016), arXiv:1410.2180.
- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez**: A characterization of weak Hopf (co)quasigroups, Mediterranean Journal of Mathematics 13, N. 5, 3747-3764 (2016), arXiv:1506.07664.
- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez** Quasigroupoids and weak Hopf quasigroups, Journal of Algebra 568, 408-436 (2021).
- **R. González Rodríguez** Weak Hopf quasigroups and matched pairs of quasigroupoids (preprint) (2022).

Weak Hopf quasigroups

Definition

A weak non associative bialgebra H in C is a unital magma (H, η_H, μ_H) and a coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(a2) $$\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$$
$$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c^{-1}_{H,H} \circ \delta_H) \otimes H).$$

(a3) $$(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$$
$$= (H \otimes (\mu_H \circ c^{-1}_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)).$$

- If the product μ_H is associative, the previous notion is the one of weak bialgebra in a braided monoidal category.
- If the product μ_H is associative and ε_H and δ_H are algebra morphisms, we have the notion of bialgebra in a braided monoidal category.
- If the ε_H and δ_H are morphisms of unital magmas, we have the notion of non associative bialgebra in a braided monoidal category.

Definition

A weak Hopf quasigroup H in C is a weak non associative bialgebra for which exists an endomorphism $\lambda_H : H \to H$ in C (called the antipode of H) such that, if we denote by Π_H^L (target morphism) and by Π_H^R (source morphism) the morphisms

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

The following equalities hold:

(b-1) $\Pi_H^L = id_H * \lambda_H$.

(b-2) $\Pi_H^R = \lambda_H * id_H$.

(b-3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H$.

(b-4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H)$.

(b-5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H)$.

(b-6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L)$.

(b-7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R)$.

- If the product μ_H is associative, the previous notion is the one of weak braided Hopf algebra introduced in

  J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, Weak braided Hopf algebras, Indiana University Mathematics Journal 57, No. 5, 2423-2458 (2008).

- If the product μ_H is associative and $\mathsf{C} = {}_{\mathbb{K}}\mathsf{Vect}$, weak Hopf quasigroups are weak Hopf algebras as was defined by:

  G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras, I. Integral theory and C^*-structure, J. Algebra 221, 385-438 (1999).

- If the product μ_H is associative and ε_H and δ_H are morphisms of algebras, we have the notion of Hopf algebra in a braided monoidal category.
- If ε_H and δ_H are morphisms of unital magmas and $\mathsf{C} = {}_{\mathbb{K}}\mathsf{Vect}$, we get the notion of Hopf quasigroup defined in:

  J. Klim, S. Majid, Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323, 3067-3110 (2010).

Definition

A Hopf quasigroup H in C is a unital magma (H, η_H, μ_H) and a coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(c1) The counit ε_H and the coproduct δ_H are morphisms of unital magmas.

(c2) There exists $\lambda_H : H \rightarrow H$ in C (called the antipode of H) such that:

(c2-1) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)$.

(c2-2) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H)$.

Example

A bicategory $\mathcal{B}$ consists of :

- A collection $\mathcal{B}_0$, whose elements x are called 0-cells.
- For each x, $y \in \mathcal{B}_0$, a category $\mathcal{B}(x,y)$ whose objects $f : x \to y$ are called 1-cells and whose morphisms $\alpha : f \Rightarrow g$ are called 2-cells. The composition of 2-cells is called the vertical composition of 2-cells

  If $f : x \to y$ is a 1-cell in $\mathcal{B}(x,y)$, x is called the source of f, represented by $s(f)$, and y is called the target of f, denoted by $t(f)$.
- For each $x \in \mathcal{B}_0$, an object $1_x \in \mathcal{B}(x,x)$, called the identity of x; and for each $x, y, z \in \mathcal{B}_0$, a functor

$$\mathcal{B}(y,z) \times \mathcal{B}(x,y) \to \mathcal{B}(x,z)$$

  which on objects is called the 1-cell composition $(g,f) \mapsto g \circ f$, and on arrows is called horizontal composition of 2-cells:

$$f, f' \in \mathcal{B}(x,y), \;\; g, g' \in \mathcal{B}(y,z), \; \alpha : f \Rightarrow f', \; \beta : g \Rightarrow g'$$

$$(\beta, \alpha) \mapsto \beta \bullet \alpha : g \circ f \Rightarrow g' \circ f'$$

- For each $f \in \mathcal{B}(x, y)$, $g \in \mathcal{B}(y, z)$, $h \in \mathcal{B}(z, w)$, an associative isomorphisms

$$\xi_{h,g,f} : (h \circ g) \circ f \Rightarrow h \circ (g \circ f);$$

and for each 1-cell f, unit isomorphisms

$$l_f : 1_{t(f)} \circ f \Rightarrow f, \quad r_f : f \circ 1_{s(f)} \Rightarrow f,$$

satisfying the following coherence axioms:

The morphisms $\xi_{h,g,f}$, l_f and r_f are natural.
Pentagon axiom: $\xi_{k,h,g\circ f} \circ \xi_{k\circ h,g,f} = (id_k \bullet \xi_{h,g,f}) \circ \xi_{k,h\circ g,f} \circ (\xi_{k,h,g} \bullet id_f)$.
Triangle axiom: $r_g \bullet id_f = (id_g \bullet l_f) \circ \xi_{g,1_{t(f)},f}$.

A bicategory is normal if the unit isomorphisms

$$l_f : 1_{t(f)} \circ f \Rightarrow f, \quad r_f : f \circ 1_{s(f)} \Rightarrow f,$$

are identities. Every bicategory is biequivalent to a normal one.

A 1-cell f is called an equivalence if there exists a 1-cell $g : t(f) \to s(f)$ and two isomorphisms $g \circ f \Rightarrow 1_{s(f)}$, $f \circ g \Rightarrow 1_{t(f)}$. In this case we will say that $g \in Inv(f)$ and, equivalently, $f \in Inv(g)$.

A bigroupoid is a bicategory where every 1-cell is an equivalence and every 2-cell is an isomorphism.

We will say that a bigroupoid $\mathcal{B}$ is finite if the collection of 0-cells $\mathcal{B}_0$ is a finite set and

$$\mathcal{B}(x, y)$$

is a small category for all x, y.

Let $\mathcal{B}$ be a finite normal bigroupoid and denote by $\mathcal{B}_1$ the set of 1-cells. Let $\mathbb{K}$ be a field and $\mathbb{K}\mathcal{B}$ the direct product

$$\mathbb{K}\mathcal{B} = \bigoplus_{f\in\mathcal{B}_\mathbf{1}} \mathbb{K}f.$$

The vector space $\mathbb{K}\mathcal{B}$ is a unital magma where the product of two 1-cells is equal to their 1-cell composition if the latter is defined and 0 otherwise, i.e., $g.f = g \circ f$ if $s(g) = t(f)$ and $g.f = 0$ if $s(g) \neq t(f)$. The unit element is

$$1_{\mathbb{K}\mathcal{B}} = \sum_{x\in\mathcal{B}_\mathbf{0}} 1_x.$$

Let $H = \mathbb{K}\mathcal{B}/I(\mathcal{B})$ be the quotient where $I(\mathcal{B})$ is the ideal of $\mathbb{K}\mathcal{B}$ generated by

$$h - g \circ (f \circ h),\ p - (p \circ f) \circ g,$$

with $f \in \mathcal{B}_1$, $g \in Inv(f)$, and $h, p \in \mathcal{B}_1$ such that $t(h) = s(f)$, $t(f) = s(p)$. In what follows, for any 1-cell f we denote its class in H by $[f]$.

If we define $[f]^{-1}$ by the class of $g \in Inv(f)$, we obtain that $[f]^{-1}$ is well-defined.

Therefore the vector space H with the product

$$\mu_H([g] \otimes [f]) = [g.f]$$

and the unit

$$\eta_H(1_{\mathbb{K}}) = \sum_{x \in \mathcal{B}_{\mathbf{0}}} [1_x]$$

is a unital magma.

Also, it is easy to show that H is a coalgebra with coproduct

$$\delta_H([f]) = [f] \otimes [f]$$

and counit

$$\varepsilon_H([f]) = 1_{\mathbb{K}}.$$

Moreover, we have a morphism (the antipode) $\lambda_H : H \to H$ defined by

$$\lambda_H([f]) = [f]^{-1}.$$

Then, H is a weak Hopf quasigroup.

Note that, if $|\mathcal{B}_0| = 1$ we obtain that H is a Hopf quasigroup. If $|\mathcal{B}_0| > 1$ and the product defined in H is associative we have an example of weak Hopf algebra.

Definition

Let H be a weak non associative bialgebra. We define the morphisms $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

and

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

Theorem

Let H be a weak non associative bialgebra. The morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ are idempotent.

Theorem

Let H be a weak non associative bialgebra. The following identities hold:

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L,$$

$$\Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R.$$

Remark

Let H be a weak non associative bialgebra. Any equality containing the idempotent morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$, $\overline{\Pi}_H^R$, the (co)product and the (co)unit, that holds for weak bialgebras also hols for H.

For example:

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H),$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H),$$

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H),$$

$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H),$$

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H,$$

$$\Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H,$$

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L,$$

$$(\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R,$$

Theorem

Let H be a weak non associative bialgebra. Put $H_L = Im(\Pi_H^L)$ and let $p_L : H \to H_L$ and $i_L : H_L \to H$ be the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = id_{H_L}$. Then,

$$H_L \xrightarrow{\ i_L\ } H \underset{(H\otimes \Pi_H^L)\circ \delta_H}{\overset{\delta_H}{\rightrightarrows}} H \otimes H$$

is an equalizer diagram and

$$H \otimes H \underset{\mu_H \circ (H\otimes \Pi_H^L)}{\overset{\mu_H}{\rightrightarrows}} H \xrightarrow{\ p_L\ } H_L$$

is a coequalizer diagram.

As a consequence, $(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$ is a unital magma in C. Also,

$$(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L)$$

is a coalgebra in C.

Theorem

Let H be a weak non associative bialgebra. The following identities hold:

$$\mu_H \circ ((\mu_H \circ (i_L \otimes H)) \otimes H) = \mu_H \circ (i_L \otimes \mu_H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (H \otimes i_L))) = \mu_H \circ (\mu_H \otimes i_L).$$

As a consequence, the unital magma H_L is a algebra in C.

If $H_R = Im(\Pi_H^R)$ we have the same properties and then H_R is a algebra in C.

Theorem

The antipode of a weak Hopf quasigroup H is unique and leaves the unit and the counit invariant, i.e. $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

Theorem

Let H be a weak Hopf quasigroup. The following identities hold:

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L,$$

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R.$$

Theorem

Let H be a weak Hopf quasigroup. The antipode of H is antimultiplicative and anticomultiplicative, i.e. the following equalities hold:

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H),$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H,$$

Therefore, excepting the associativity, weak Hopf quasigroups satisfy the relevant equalities that we can find in the theory of weak Hopf algebras.

Definition

Let H, H' be weak Hopf quasigroups. We will say that $f : H \to H'$ is a morphism of weak Hopf quasigroups if it is a coalgebra morphism such that

$$\Pi^R_{H'} \circ f = f \circ \Pi^R_H,$$
$$\overline{\Pi}^L_{H'} \circ f = f \circ \overline{\Pi}^L_H,$$
$$\Pi^R_{H'} \circ \Pi^L_{H'} \circ f = f \circ \Pi^R_H \circ \Pi^L_H,$$
$$f \circ \mu_H = \mu_{H'} \circ (f \otimes f) \circ \nabla_H,$$

hold, where $\nabla_H : H \otimes H \to H \otimes H$ is the idempotent morphism defined by

$$\nabla_H = (H \otimes (\mu_H \circ (\Pi^R_H \otimes H))) \circ (\delta_H \otimes H).$$

- The previous definition is similar to the one introduced in

  G. Böhm, J. Gómez-Torrecillas, E. López Centella, On the category of weak bialgebras, J. Algebra 399 (2014), 801-844.

  for weak Hopf algebras.

  With WHQ we will denote the category of weak Hopf quasigroups.

Fusion morphisms and the characterization of weak Hopf quasigroups

Now let H be a weak non associative bialgebra. Define the Ω-morphisms,

$$\Omega_L^1 = (\mu_H \otimes H) \circ (H \otimes \Pi_H^L \otimes H) \circ (H \otimes \delta_H),$$

$$\Omega_R^1 = (\mu_H \otimes H) \circ (H \otimes \Pi_H^R \otimes H) \circ (H \otimes \delta_H),$$

$$\Omega_L^2 = (H \otimes \mu_H) \circ (H \otimes \Pi_H^L \otimes H) \circ (\delta_H \otimes H),$$

$$\Omega_R^2 = (H \otimes \mu_H) \circ (H \otimes \Pi_H^R \otimes H) \circ (\delta_H \otimes H).$$

These morphisms are idempotent and, as a consequence, there exist objects $H \times_L^1 H$, $H \times_R^1 H$, $H \times_L^2 H$ and $H \times_R^2 H$ and morphisms

$$q_L^1 : H \otimes H \to H \times_L^1 H, \quad j_L^1 : H \times_L^1 H \to H \otimes H,$$

$$q_R^1 : H \otimes H \to H \times_R^1 H, \quad j_R^1 : H \times_R^1 H \to H \otimes H,$$

$$q_L^2 : H \otimes H \to H \times_L^2 H, \quad j_L^2 : H \times_L^2 H \to H \otimes H,$$

$$q_R^2 : H \otimes H \to H \times_R^2 H, \quad j_R^2 : H \times_R^2 H \to H \otimes H,$$

such that, for $\sigma \in \{L, R\}$ and $\alpha \in \{1, 2\}$,

$$j_\sigma^\alpha \circ q_\sigma^\alpha = \Omega_\sigma^\alpha, \quad q_\sigma^\alpha \circ j_\sigma^\alpha = id_{H \times_\sigma^\alpha H}.$$

Theorem

Let H be a weak non associative bialgebra. Then we have that:

(i) The diagrams

$$H \otimes H_L \otimes H \; \underset{H\otimes(\mu_H \circ (i_L \otimes H))}{\overset{(\mu_H \circ (H \otimes i_L)) \otimes H}{\rightrightarrows}} \; H \otimes H \xrightarrow{q_L^1} H \times_L^1 H$$

and

$$H \otimes H_R \otimes H \; \underset{H\otimes(\mu_H \circ (i_R \otimes H))}{\overset{(\mu_H \circ (H \otimes i_R)) \otimes H}{\rightrightarrows}} \; H \otimes H \xrightarrow{q_R^2} H \times_R^2 H$$

are coequalizer diagrams.

(ii) The diagrams

$$H \times_L^2 H \xrightarrow{j_L^2} H \otimes H \; \underset{H \otimes ((p_L \otimes H) \circ \delta_H)}{\overset{((H \otimes p_L) \circ \delta_H) \otimes H}{\rightrightarrows}} \; H \otimes H_L \otimes H$$

and

$$H \times_R^1 H \xrightarrow{j_R^1} H \otimes H \; \underset{H \otimes ((p_R \otimes H) \circ \delta_H)}{\overset{((H \otimes p_R) \circ \delta_H) \otimes H}{\rightrightarrows}} \; H \otimes H_L \otimes H$$

are equalizer diagrams.

Definition

Let H be a unital magma. We say that a morphism $\phi : H \otimes H \to H \otimes H$ is:

(i) Almost left H-linear, if $\phi = (\mu_H \otimes H) \circ (H \otimes \phi) \circ (H \otimes \eta_H \otimes H)$.

(ii) Almost right H-linear, if $\phi = (H \otimes \mu_H) \circ (\phi \otimes H) \circ (H \otimes \eta_H \otimes H)$.

By dualization, if H is a counital comagma, we will say that a morphism ϕ is almost left H-colinear if

$$\phi = (H \otimes \varepsilon_H \otimes H) \circ (H \otimes \phi) \circ (\delta_H \otimes H),$$

and almost right H-colinear if

$$\phi = (H \otimes \varepsilon_H \otimes H) \circ (\phi \otimes H) \circ (H \otimes \delta_H).$$

Theorem

Let H be a unital magma and a counital comagma. The following assertions hold.

(i) The right Galois morphism, defined as

$$\beta = (\mu_H \otimes H) \circ (H \otimes \delta_H),$$

is almost left H-linear and almost right H-colinear.

(ii) The left Galois morphism, defined as

$$\gamma = (H \otimes \mu_H) \circ (\delta_H \otimes H),$$

is almost right H-linear and almost left H-colinear.

(iii) The morphisms Ω_L^1 and Ω_R^1 are almost left H-linear and almost right H-colinear.

(iv) The morphisms Ω_L^2 and Ω_R^2 are almost right H-linear and almost left H-colinear.

From now on, the morphisms β and γ will be called fusion morphisms.

Note that, if H is a weak Hopf quasigroup, we can express the Ω-morphisms as compositions of the fusion morphisms in the following way

$$\Omega_L^1 = \overline{\beta} \circ \beta, \quad \Omega_R^1 = \beta \circ \overline{\beta}, \quad \Omega_L^2 = \gamma \circ \overline{\gamma}, \quad \Omega_R^2 = \overline{\gamma} \circ \gamma,$$

where

$$\overline{\beta} = (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H)$$

and

$$\overline{\gamma} = (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H).$$

Moreover, if the weak Hopf quasigroup H is a Hopf quasigroup,

$$\Pi_H^L = \Pi_H^R = \overline{\Pi}_H^L = \overline{\Pi}_H^R = \varepsilon_H \otimes \eta_H$$

and then the Ω-morphisms are identities. As a consequence we have that in this case the fusion morphisms β and γ are isomorphisms with inverses $\overline{\beta}$ and $\overline{\gamma}$, respectively.

Theorem

Let H be a weak non associative bialgebra. The following assertions are equivalent:

(i) H is a weak Hopf quasigroup.

(ii) The morphisms
$$f = q_R^1 \circ \beta \circ j_L^1 : H \times_L^1 H \to H \times_R^1 H$$
and
$$g = q_L^2 \circ \gamma \circ j_R^2 : H \times_R^2 H \to H \times_L^2 H$$
are isomorphisms, the morphism $j_L^1 \circ f^{-1} \circ q_R^1$ is almost left H-linear and $j_R^2 \circ g^{-1} \circ q_L^2$ is almost right H-linear.

Proof

(ii)$\Rightarrow$ (i)
$$\lambda_H = (H \otimes \varepsilon_H) \circ j_R^1 \circ f^{-1} \circ q_R^1 \circ (\eta_H \otimes H)$$

Corollary

Let H be a unital magma and coalgebra such that ε_H and δ_H are morphisms of unital magmas. Then H is a Hopf quasigroup if and only if the fusion morphisms β and γ are isomorphisms and they have almost left H-linear and almost right H-linear inverses, respectively.

- This result (called the first fundamental theorem for Hopf (co)quasigroups) was proved by T. Brzezinski for Hopf quasigroups in ${}_{\mathbb{K}}$Vect in:

T. Brzeziński, Hopf modules and the fundamental theorem for Hopf quasigroups, Internat. Elec. J. Algebra 8, 114-128 (2010).

Corollary

Let H be a weak bialgebra. The following assertions are equivalent.

(i) H is a weak Hopf algebra.

(ii) The morphism

$$f = q_R^1 \circ \beta \circ j_L^1 : H \times_L^1 H \to H \times_R^1 H$$

is an isomorphism.

(ii) The morphism

$$g = q_L^2 \circ \gamma \circ j_R^2 : H \times_R^2 H \to H \times_L^2 H$$

is an isomorphism.

- This result was proved by P. Schauenburg for weak Hopf algebras in ${}_{\mathbb{K}}$Vect in:

P. Schauenburg, Weak Hopf algebras and quantum groupoids, Noncommutative geometry and quantum groups (Warsaw, 2001), 171-188, Polish Acad. Sci., Warsaw, (2003).

Quasigroupoids

Definition

A quasigroupoid A is an ordered pair of sets $A = (A_0, A_1)$ such that:

(d1) There exist maps $s_A : A_1 \to A_0$, $t_A : A_1 \to A_0$, and $id_A : A_0 \to A_1$, called source, target and identity, respectively, satisfying

$$s_A(id_A(x)) = t_A(id_A(x)) = x, \quad \forall\, x \in A_0.$$

(d2) There exist a map, called product of A,

$$\bullet : A_1 \;{}_{s_A}\!\times_{t_A} A_1 = \{(a, b) \in A_1 \times A_1 \;;\; s_A(a) = t_A(b)\} \to A_1,$$

defined by $\bullet(a, b) = a \bullet b$ and a map $\lambda_A : A_1 \to A_1$, called the inverse map, such that:

(d2-1) For each $a \in A_1$,
$$id_A(t_A(a)) \bullet a = a = a \bullet id_A(s_A(a)).$$

(d2-2) For all $(a, b) \in A_1 \;{}_{s_A}\!\times_{t_A} A_1$,
$$s_A(a \bullet b) = s_A(b), \quad t_A(a \bullet b) = t_A(a).$$

(d2-3) For all $(a, b) \in A_1 \;{}_{s_A}\!\times_{t_A} A_1$, $(\lambda_A(a), a \bullet b)$ and $(a \bullet b, \lambda_A(b))$ are in $A_1 \;{}_{s_A}\!\times_{t_A} A_1$ and
$$\lambda_A(a) \bullet (a \bullet b) = b, \quad (a \bullet b) \bullet \lambda_A(b) = a.$$

- The set A_0 wil be called the base of A. We will say that a quasigroupoid A is finite if its base is a finite set.
- Note that a finite quasigroupoid where $|A_0| = 1$ is an I.P. loop or a quasigroup in the sense of J. Klim and S. Majid:

Definition

A quasigroup is a set A with a product, identity e_A and with the property that for each $a \in A$ there is $a^{-1} \in A$ such that

$$a^{-1}(ab) = b, \quad (ba)a^{-1} = b$$

for all $b \in A$.

- If the product of a quasigroupoid A is associative, A is a groupoid, i.e., a category where every morphism is an isomorphism.

- A quasigroupoid is an inverse semiloopoid satisfying the unities associativity assumption. These notions were introduced by J. Grabowski in

  J. Grabowski. An introduction to loopoids, Comment. Math. Univ. Carolin. 57 (2016), 515-526.

  If A is a quasigroupoid, the following equalities:

$$
\begin{aligned}
s_{\mathsf{A}}(\lambda_{\mathsf{A}}(a)) &= t_{\mathsf{A}}(a),\\
t_{\mathsf{A}}(\lambda_{\mathsf{A}}(a)) &= s_{\mathsf{A}}(a),\\
\lambda_{\mathsf{A}}(a) \bullet a &= id_{\mathsf{A}}(s_{\mathsf{A}}(a)),\\
a \bullet \lambda_{\mathsf{A}}(a) &= id_{\mathsf{A}}(t_{\mathsf{A}}(a)),\\
\lambda_{\mathsf{A}}(\lambda_{\mathsf{A}}(a)) &= a,\\
\lambda_{\mathsf{A}}(a \bullet b) &= \lambda_{\mathsf{A}}(b) \bullet \lambda_{\mathsf{A}}(a),
\end{aligned}
$$

  hold for all $a \in \mathsf{A}_1$ and $(a, b) \in \mathsf{A}_1 \ {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1$.

Definition

Let A, A′ be quasigroupoids. A morphism $\Gamma : A \to A'$ between A and A′ is a pair of maps $\Gamma = (\Gamma_0, \Gamma_1)$, $\Gamma_0 : A_0 \to A'_0$, $\Gamma_1 : A_1 \to A'_1$, such that

(e1) $\Gamma_0 \circ s_A = s_{A'} \circ \Gamma_1$,

(e2) $\Gamma_0 \circ t_A = t_{A'} \circ \Gamma_1$,

(e3) For all $x \in A_0$, $\Gamma_1(id_A(x)) = id_{A'}(\Gamma_0(x))$,

(e4) For all $(a, b) \in A_1 \ {}_{s_A}\times_{t_A} A_1$, $\Gamma_1(a \bullet b) = \Gamma_1(a) \bullet' \Gamma_1(b)$.

The obvious composition of quasigroupoid morphisms is a quasigroupoid morphism. Then with QGPD we will denote the category whose objects are quasigroupoids and whose morphisms are morphisms of quasigroupoids. With $\overline{\text{QGPD}}$ we will denote the full subcategory of QGPD whose objects are finite quasigroupoids.

Example

Let A be a quasigroup (in the sense of J. Klim and S. Majid) with product $\cdot$ and let X be a set. Assume that there exists a map

$$\psi_X : A \times X \to X$$

satisfying the following two conditions:

$$\psi_X(e_A, x) = x,$$

$$\psi_X(a \cdot b, x) = \psi_X(a, \psi_X(b, x)),$$

for all $x \in X$ and $a, b \in A$.

The quasigroupoid $\mathsf{B} = (\mathsf{B}_0, \mathsf{B}_1)$ associated to the action ψ_X is defined by the sets $\mathsf{B}_0 = X$, $\mathsf{B}_1 = A \times X$ and maps

$$s_{\mathsf{B}} : \mathsf{B}_1 \to \mathsf{B}_0, \quad s_{\mathsf{B}}(a, x) = x,$$

$$t_{\mathsf{B}} : \mathsf{B}_1 \to \mathsf{B}_0, \quad t_{\mathsf{B}}(a, x) = \psi_X(a, x),$$

$$id_{\mathsf{B}} : \mathsf{B}_0 \to \mathsf{B}_1, \quad id_{\mathsf{B}}(x) = (e_A, x).$$

Then,

$$\mathsf{B}_1 \,{}_{s_{\mathsf{B}}}\!\times_{t_{\mathsf{B}}} \mathsf{B}_1 = \{((a, x), (b, y),) \in \mathsf{B}_1 \times \mathsf{B}_1 \;/\; \psi_X(b, y) = x\}$$

and the product is defined by

$$(a, x) \star (b, y) = (a \cdot b, y).$$

The inverse map $\lambda_{\mathsf{B}} : \mathsf{B}_1 \to \mathsf{B}_1$ is

$$\lambda_{\mathsf{B}}(a, x) = (a^{-1}, \psi_X(a, x)).$$

- As was proved in
  Alonso Álvarez, J.N., Fernández Vilaboa, J.M., González Rodríguez, R. Quasigroupoids and weak Hopf quasigroups Journal of Algebra 568, 408-436 (2021)
  examples of this kind can be obtained by working with Moufang loops of small order and the 4-dimensional Taft Hopf algebra.

Example

Let A be a quasigroup and let X be a set. Denote by T the set

$$T = \{(a,x,y) \ / \ a \in A, \ x,y \in X\}.$$

The quasigroupoid $\mathsf{T} = (\mathsf{T}_0, \mathsf{T}_1)$ associated to T is defined by the sets

$$\mathsf{T}_0 = \{(e_A,x,x) \in T\}, \quad \mathsf{T}_1 = T$$

and maps

$$s_{\mathsf{T}} : \mathsf{T}_1 \to \mathsf{T}_0, \quad s_{\mathsf{T}}(a,x,y) = (e_A,y,y),$$

$$t_{\mathsf{T}} : \mathsf{T}_1 \to \mathsf{T}_0, \quad t_{\mathsf{T}}(a,x,y) = (e_A,x,x),$$

$$id_{\mathsf{T}} : \mathsf{T}_0 \to \mathsf{T}_1, \quad id_{\mathsf{T}}(e_A,x,x) = (e_A,x,x).$$

Then,

$$\mathsf{T}_1 \ {}_{s_{\mathsf{T}}}\!\times_{t_{\mathsf{T}}} \mathsf{T}_1 = \{((a,x,y),(b,y,r)) \in \mathsf{T}_1 \times \mathsf{T}_1\}$$

and the product is defined by

$$(a,x,y) \star (b,y,r) = (a \cdot b, x, r).$$

The inverse map $\lambda_{\mathsf{T}} : \mathsf{T}_1 \to \mathsf{T}_1$ is $\lambda_{\mathsf{B}}(a,x,y) = (a^{-1},y,x)$.

Example

Let A be a quasigroupoid, let P be a set and let $\pi : P \to \mathrm{A}_0$ be a surjective map. The quasigroupoid $\mathrm{P(A)}^{\pi} = (\mathrm{P(A)}^{\pi}_0, \mathrm{P(A)}^{\pi}_1)$ is defined by the sets $\mathrm{P(A)}^{\pi}_0 = P$,

$$\mathrm{P(A)}^{\pi}_1 = \{(p, a, q) \;/\; (p, a, q) \in P \times \mathrm{A}_1 \times P, \; \pi(p) = t_{\mathrm{A}}(a), \; \pi(q) = s_{\mathrm{A}}(a)\}$$

and maps

$$s_{\mathrm{P(A)}^{\pi}} : \mathrm{P(A)}^{\pi}_1 \to \mathrm{P(A)}^{\pi}_0, \quad s_{\mathrm{P(A)}^{\pi}}(p, a, q) = q,$$
$$t_{\mathrm{P(A)}^{\pi}} : \mathrm{P(A)}^{\pi}_1 \to \mathrm{P(A)}^{\pi}_0, \quad t_{\mathrm{P(A)}^{\pi}}(p, a, q) = p,$$
$$id_{\mathrm{P(A)}^{\pi}} : \mathrm{P(A)}^{\pi}_0 \to \mathrm{P(A)}^{\pi}_1, \quad id_{\mathrm{P(A)}^{\pi}}(p, a, q) = (p, id_{\pi(p)}, p).$$

Then,

$$\mathrm{P(A)}^{\pi}_1 \;{}_{s_{\mathbf{P(A)}^{\pi}}}\!\times_{t_{\mathbf{P(A)}^{\pi}}} \mathrm{P(A)}^{\pi}{}_1 = \{((p, a, q), (q, b, n)) \in \mathrm{P(A)}^{\pi}{}_1 \times \mathrm{P(A)}^{\pi}{}_1\}$$

and the product is defined by

$$(p, a, q) \star (q, b, n) = (p, a \bullet b, n).$$

The inverse map $\lambda_{\mathrm{P(A)}^{\pi}} : \mathrm{P(A)}^{\pi}_1 \to \mathrm{P(A)}^{\pi}_1$ is

$$\lambda_{\mathrm{P(A)}^{\pi}}(p, a, q) = (q, \lambda_{\mathrm{A}}(a), p).$$

Theorem

Let $\mathbb{K}$ be a field and let $\mathsf{A} = (\mathsf{A}_0, \mathsf{A}_1)$ be a finite quasigroupoid. The quasigroupoid magma $\mathbb{K}[\mathsf{A}]$ defined by

$$\mathbb{K}[\mathsf{A}] = \bigoplus_{a \in \mathsf{A}_1} \mathbb{K}a$$

is a cocommutative weak Hopf quasigroup with unit $1 = \sum_{x \in \mathsf{A}_0} id_{\mathsf{A}}(x)$, product

$$\mu_{\mathbb{K}[\mathsf{A}]}(a \otimes b) = \begin{cases} a \bullet b, & \text{if } (a,b) \in \mathsf{A}_1 \; {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1, \\ 0, & \text{if } (a,b) \notin \mathsf{A}_1 \; {}_{s_{\mathsf{A}}}\!\times_{t_{\mathsf{A}}} \mathsf{A}_1, \end{cases}$$

counit $\varepsilon_{\mathbb{K}[\mathsf{A}]}(a) = 1_{\mathbb{K}}$, coproduct $\delta_{\mathbb{K}[\mathsf{A}]}(a) = a \otimes a$ and antipode $\lambda_{\mathbb{K}[\mathsf{A}]}(a) = \lambda_{\mathsf{A}}(a)$ on the basis elements.

Theorem

Let $\mathbb{K}$ be a field and let $\Gamma = (\Gamma_0, \Gamma_1) : A \to A'$ be a morphism of finite quasigroupoids. Then the linear extension $\mathbb{K}[\Gamma_1] : \mathbb{K}[A] \to \mathbb{K}[A']$, $\mathbb{K}[\Gamma_1](a) = \Gamma_1(a)$, is a morphism of weak Hopf quasigroups that we will denote by $\mathbb{K}[\Gamma]$.

Theorem

There exists a functor, called the WHQ-functor,

$$F : \overline{\text{QGPD}} \to \text{WHQ}$$

defined on objects by $F(A) = \mathbb{K}[A]$ and on morphisms by $F(\Gamma) = \mathbb{K}[\Gamma]$.

Categorical equivalences

In this section we will assume that $\mathsf{C} = {}_{\mathbb{K}}\mathsf{Vect}$.

Theorem

Let H be a weak Hopf quasigroup in C. The algebras H_L and H_R are separable, finite dimensional and semisimple.

Definition

Let H be a weak Hopf quasigroup and $\mathrm{G}(H)$ its set of group-like elements as a coalgebra (coalgebra), i. e., $\mathrm{G}(H)$ is the set of $\mathrm{g} \in H$ such that $\delta_H(\mathrm{g}) = \mathrm{g} \otimes \mathrm{g}$ and $\varepsilon_H(\mathrm{g}) = 1_{\mathbb{K}}$.

Theorem

Let H be a weak Hopf quasigroup and consider $t \in H$ such that $\delta_H(t) = t \otimes t$. Then the following equalities hold:

(i) $\Pi_H^L(t)t = \overline{\Pi}_H^L(t)t = t\Pi_H^R(t) = t\overline{\Pi}_H^R(t) = t$.

(ii) The elements $\Pi_H^L(t)$, $\overline{\Pi}_H^L(t)$, $\Pi_H^R(t)$ and $\overline{\Pi}_H^R(t)$ are idempotent.

(iii) $\delta_H(\Pi_H^L(t)) = \Pi_H^L(t) \otimes \Pi_H^L(t), \quad \delta_H(\Pi_H^R(t)) = \Pi_H^R(t) \otimes \Pi_H^R(t)$.

(iv) $(\Pi_H^L \circ \Pi_H^R)(t) = \Pi_H^R(t), \quad (\Pi_H^R \circ \Pi_H^L)(t) = \Pi_H^L(t)$.

(v) $\Pi_H^L(t) = \overline{\Pi}_H^L(t), \quad \Pi_H^R(t) = \overline{\Pi}_H^R(t)$.

(vi) $\delta_H(\overline{\Pi}_H^L(t)) = \overline{\Pi}_H^L(t) \otimes \overline{\Pi}_H^L(t), \quad \delta_H(\overline{\Pi}_H^R(t)) = \overline{\Pi}_H^R(t) \otimes \overline{\Pi}_H^R(t)$.

(vii) $(\Pi_H^L \circ \lambda_H)(t) = \Pi_H^R(t) = (\lambda_H \circ \Pi_H^R)(t), \quad (\Pi_H^R \circ \lambda_H)(t) = \Pi_H^L(t) = (\lambda_H \circ \Pi_H^L)(t)$.

(viii) $(\lambda_H^2 \circ \Pi_H^L)(t) = \Pi_H^L(t) = (\Pi_H^L \circ \lambda_H^2)(t), \quad (\lambda_H^2 \circ \Pi_H^R)(t) = \Pi_H^R(t) = (\Pi_H^R \circ \lambda_H^2)(t)$.

(ix) $\lambda_H^2(t) = t$.

Corollary

Let H be a weak Hopf quasigroup. If $\mathrm{g} \in H$ is a grouplike element of H, then so are $\Pi_H^L(\mathrm{g})$, $\Pi_H^R(\mathrm{g})$ and $\lambda_H(\mathrm{g})$.

Theorem

Let H be a weak Hopf quasigroup. The group-like elements of H are linearly independent and the subspace H_R is a finite dimensional algebra. Therefore, the cardinality of

$$\mathrm{G}(H) \cap H_R$$

is finite.

Theorem

Let H be a weak Hopf quasigroup. The ordered pair of sets $\mathcal{T}(H) = (\mathcal{T}(H)_0, \mathcal{T}(H)_1)$, where

$$\mathcal{T}(H)_0 = \mathrm{G}(H) \cap H_R, \quad \mathcal{T}(H)_1 = \mathrm{G}(H),$$

is a finite quasigroupoid where the source, target and identity morphisms

$$\mathfrak{s} : \mathcal{T}(H)_1 \to \mathcal{T}(H)_0, \quad \mathfrak{t} : \mathcal{T}(H)_1 \to \mathcal{T}(H)_0, \quad \mathfrak{id} : \mathcal{T}(H)_0 \to \mathcal{T}(H)_1,$$

are defined by

$$\mathfrak{s}(\mathrm{g}) = \Pi_H^R(\mathrm{g}), \quad \mathfrak{t}(\mathrm{g}) = \Pi_H^L(\mathrm{g}), \quad \mathfrak{id}(\mathrm{r}) = \mathrm{r},$$

the product

$$\star : \mathcal{T}(H)_1 \; {}_{\mathfrak{s}}\times_{\mathfrak{t}} \; \mathcal{T}(H)_1 \to \mathcal{T}(H)_1$$

is defined by $\mathrm{h} \star \mathrm{g} = \mu_H(\mathrm{h} \otimes \mathrm{g})$, and the inverse map $\lambda : \mathcal{T}(H)_1 \to \mathcal{T}(H)_1$ is

$$\lambda(\mathrm{g}) = \lambda_H(\mathrm{g}).$$

Theorem

Let $f : H \to H'$ be a morphism of weak Hopf quasigroups. The pair

$$\mathcal{T}(f) = (\mathcal{T}(f)_0, \mathcal{T}(f)_1),$$

where

$$\mathcal{T}(f)_0 : \mathcal{T}(H)_0 \to \mathcal{T}(H')_0, \quad \mathcal{T}(f)_1 : \mathcal{T}(H)_1 \to \mathcal{T}(H')_1$$

are the maps defined by $\mathcal{T}(f)_0(\mathrm{r}) = f(\mathrm{r})$ and $\mathcal{T}(f)_1(\mathrm{g}) = f(\mathrm{g})$, is a morphism of quasigroupoids between $\mathcal{T}(H)$ and $\mathcal{T}(H')$.

Theorem

There exists a functor, called the QGPD-functor,

$$\mathrm{L} : \mathrm{WHQ} \to \overline{\mathrm{QGPD}}$$

defined on objects by $\mathrm{L}(H) = \mathcal{T}(H)$ and on morphisms by $\mathrm{L}(f) = \mathcal{T}(f)$.

Theorem

The WHQ-functor is left adjoint of the QGPD-functor.

Theorem

The functors F and L induce an equivalence between the category $\overline{\mathrm{QGPD}}$ and the full subcategory of WHQ of all pointed cosemisimple weak Hopf quasigroups.

Corollary

If $\mathbb{K}$ is algebraically closed, the functors F and L induce an equivalence between the category $\overline{\mathrm{QGPD}}$ and the full subcategory of WHQ of all cocommutative cosemisimple weak Hopf quasigroups.

- In the associative setting (replace quasigroupoids by small groupoids with finitely many objects), the previous results are the ones proved in

  G. Böhm, J. Gómez-Torrecillas, E. López Centella, On the category of weak bialgebras, J. Algebra 399 (2014), 801-844.

  for weak Hopf algebras.

Theorem

The category of quasigroups is equivalent to the one of pointed cosemisimple Hopf quasigroups.

Corollary

If $\mathbb{K}$ is algebraically closed, the category of quasigroups is equivalent to the one of cocommutative cosemisimple Hopf quasigroups.

Thank you