Hopf trusses an related structures in a monoidal setting

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Preliminaries and notations

Hopf trusses an related structures in a monoidal setting

Throughout this talk C denotes a strict braided monoidal category with tensor product $\otimes$, unit object K and braiding c.

Recall that a monoidal category is a category C together with a functor

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$$

called tensor product, an object K of C, called the unit object, and families of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in C, called associativity, right unit and left unit constraints, respectively, satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where for each object X in C, id_X denotes the identity morphism of X.

A monoidal category is called strict if the constraints of the previous paragraph are identities.

It is a well-known fact that every non-strict monoidal category is monoidal equivalent to a strict one. This lets us to treat monoidal categories as if they were strict and, as a consequence, the results proved in a strict setting hold for every non-strict monoidal category.

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For simplicity of notation, given objects M, N, P in C and a morphism $f : M \to N$, we will write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.
A braiding for a strict monoidal category C is a natural family of isomorphisms

$$c_{M,N} : M \otimes N \to N \otimes M$$

subject to the conditions

$$c_{M,N\otimes P} = (N \otimes c_{M,P}) \circ (c_{M,N} \otimes P), \quad c_{M\otimes N,P} = (c_{M,P} \otimes N) \circ (M \otimes c_{N,P})$$

for all $M, N, P \in$ C.
If

$$c_{N,M} \circ c_{M,N} = id_{M\otimes N}$$

for all M, N in C, we will say that C is symmetric.
Then the results presented in this talk hold in

- Set, the category of sets.
- $\mathbb{F}$-Vect, the category of vector spaces over a field $\mathbb{F}$.
- ${}_{\mathbf{R}}$Mod, the category of left modules over a commutative ring R.
- Rep(G), the category of representations of a group G.
- sVect, the category of super-vector spaces.
- $\mathbb{B}$, the braid category.
- ${}_{\mathbf{H}}$Mod, the category of left H-modules for a quasitriangular Hopf algebra.
- ${}^{\mathbf{H}}_{\mathbf{H}}$YD, the category of left Yetter-Drinfeld modules over a Hopf algebra such that the antipode is an isomorphism.

Definition

An algebra in C is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in C and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in C such that

$$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A), \quad \mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$$

hold.

Definition

Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, a morphism $f : A \to B$ in C is an algebra morphism if

$$f \circ \eta_A = \eta_B, \quad \mu_B \circ (f \otimes f) = f \circ \mu_A$$

hold.

If A, B are algebras in C, the tensor product $A \otimes B$ is also an algebra in C where

$$\eta_{A\otimes B} = \eta_A \otimes \eta_B, \quad \mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B).$$

Definition

A coalgebra in C is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in C and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in C such that

$$(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D, \quad (\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$$

hold.

Definition

If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, a morphism $f : D \to E$ in C is a coalgebra morphism if

$$\varepsilon_E \circ f = \varepsilon_D, \quad (f \otimes f) \circ \delta_D = \delta_E \circ f$$

hold.

Given D, E coalgebras in C, the tensor product $D \otimes E$ is a coalgebra in C where

$$\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E, \quad \delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E).$$

Example

In the category of vector spaces over a field $\mathbb{F}$ we can find interesting examples of coalgebras. For example, if S is a set, with $\mathbb{F}[S]$ we will denote the free $\mathbb{F}$-vector space on S, i.e.,

$$\mathbb{F}[S] = \bigoplus_{s \in S} \mathbb{F}s.$$

This vector space has a coalgebra structure determined by

$$\varepsilon_{\mathbb{F}[S]}(s) = 1_{\mathbb{F}}, \quad \delta_{\mathbb{F}[S]}(s) = s \otimes s.$$

Definition

Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra in C. We will say that a morphism $g : K \to D$ is a grouplike morphism if satisfy $\delta_D \circ g = g \otimes g, \quad \varepsilon_D \circ g = id_K$.

Definition

Let $(D, \varepsilon_D, \delta_D)$ be a coalgebra in $\mathbb{F}$-Vect. A grouplike element c of D is a $c \in D$ such that the linear map $g_c : \mathbb{F} \to D$ defined by $g_c(1_{\mathbb{F}}) = c$ is a grouplike morphism in $\mathbb{F}$-Vect.

In the following we will denote by $\mathrm{G}(D)$ the set of grouplike elements of D and $\mathrm{G}(D)$ is a subcoalgebra of D.
If S is a set, the coalgebra $\mathbb{F}[S]$ is called the grouplike coalgebra of S and satisfies

$$\mathrm{G}(\mathbb{F}[S]) = S.$$

Definition

A pointed coalgebra in $\mathbb{F}$-Vect is a coalgebra D whose simple subcoalgebras are one-dimensional.
Then, D is pointed if and only if its coradical D_0 (the sum of the simple subcoalgebras of D) is the grouplike coalgebra of $\mathrm{G}(D)$, i.e., $D_0 = \mathbb{F}[\mathrm{G}(D)]$.

Definition

We will say that the coalgebra D is cosemisimple if $D = D_0$.

Therefore, if D is pointed cosemisimple, $D = \mathbb{F}[\mathrm{G}(D)]$. On the other hand, if G is a group and $D = \mathbb{F}[G]$, we have that D is pointed and cosemisimple.

Definition

Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra and let $A = (A, \eta_A, \mu_A)$ be an algebra. By

$$\mathcal{H}(D, A)$$

we denote the morphisms $f : D \to A$ in C. With the convolution operation

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_D,$$

$\mathcal{H}(D, A)$ is an monoid where the unit element is $\eta_A \circ \varepsilon_D = \varepsilon_D \otimes \eta_A$.

We will say that $f : D \to A$ is convolution invertible if there exists $f^{-1} : D \to A$ such that

$$f * f^{-1} = f^{-1} * f = \varepsilon_D \otimes \eta_A.$$

Definition

Let A be an algebra. The pair (M, φ_M) is a left A-module if M is an object in C and $\varphi_M : A \otimes M \to M$ is a morphism in C satisfying

$$\varphi_M \circ (\eta_A \otimes M) = id_M, \quad \varphi_M \circ (A \otimes \varphi_M) = \varphi_M \circ (\mu_A \otimes M).$$

Given two left A-modules (M, φ_M) and (N, φ_N), $f : M \to N$ is a morphism of left A-modules if $\varphi_N \circ (A \otimes f) = f \circ \varphi_M$.
Then left A-modules with morphisms of left A-modules form a category that we will denote by ${}_{\text{A}}\text{Mod}$.

Definition

Let B an object in C such that there exists an associative product $\mu_B : B \otimes B \to B$. We will say that $(M, \phi_M : B \otimes M \to M)$ is a non-unital left B-module if

$$\phi_M \circ (B \otimes \phi_M) = \phi_M \circ (\mu_B \otimes M).$$

A morphism between non-unital left B-modules is a left B-linear morphism as in the case of morphisms for modules over an algebra. Then non-unital left B-modules form a category that we will denote by ${}_{\text{B}}\text{mod}$.

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Definition

A non-unital bialgebra in the category C is a coalgebra $(B, \varepsilon_B, \delta_B)$ with an associative product $\mu_B : B \otimes B \to B$ such that μ_B is a coalgebra morphism. Then the following identities hold:

$$\varepsilon_B \circ \mu_B = \varepsilon_B \otimes \varepsilon_B,$$

$$\delta_B \circ \mu_B = (\mu_B \otimes \mu_B) \circ \delta_{B \otimes B}.$$

A bialgebra in C is an algebra (B, η_B, μ_B) and a coalgebra $(B, \varepsilon_B, \delta_B)$ such that η_B and μ_B are coalgebra morphisms. Then,

$$\varepsilon_B \circ \eta_B = id_K, \quad \delta_B \circ \eta_B = \eta_B \otimes \eta_B$$

also hold.
A morphism between non-unital bialgebras H and B is a morphism $f : H \to B$ in C of coalgebras and multiplicative. A morphism between bialgebras H and B is a morphism $f : H \to B$ in C of algebras and coalgebras.
With the composition of morphisms in C we can define a category whose objects are non-unital bialgebras (bialgebras) and whose morphisms are morphisms of non-unital bialgebras (bialgebras). We denote this category by bialg (Bialg).

Definition

Let B a non-unital bialgebra and let A be an algebra in C. We will say that (A, ϕ_A) is a non-unital left B-module algebra if it is a non-unital left B-module with action $\phi_A : B \otimes A \to A$ such that

$$\phi_A \circ (B \otimes \eta_A) = \varepsilon_B \otimes \eta_A$$

and

$$\phi_A \circ (B \otimes \mu_A) = \mu_A \circ (\phi_A \otimes \phi_A) \circ (B \otimes c_{B,A} \otimes A) \circ (\delta_B \otimes A \otimes A)$$

hold.
If B is a bialgebra, we will say that (A, ϕ_A) is a left B-module algebra if (A, ϕ_A) is a left B-module and the two previous conditions hold.

Definition

Let B be a non-unital bialgebra and $D = (D, \varepsilon_D, \delta_D)$ a coalgebra in C. A pair (D, φ_D) is said to be a non-unital left B-module coalgebra if (D, φ_D) is a non-unital left B-module and the following equalities hold:

$$\varepsilon_D \circ \varphi_D = \varepsilon_B \otimes \varepsilon_D,$$

and

$$\delta_D \circ \varphi_D = (\varphi_D \otimes \varphi_D) \circ \delta_{B \otimes D}.$$

In case that B is a bialgebra, a non-unital left B-module coalgebra (D, φ_D) is said to be a left B-module coalgebra if (D, φ_D) is a left B-module.

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Definition

Let H be a bialgebra in C. If there exists a morphism $\lambda_H : H \to H$ in C, called the antipode of H, satisfying that λ_H is the inverse of id_H in $\mathcal{H}(H, H)$, i.e.,

$$id_H * \lambda_H = \eta_H \circ \varepsilon_H = \lambda_H * id_H,$$

we say that H is a Hopf algebra.

A morphism of Hopf algebras is an bialgebra morphism. We can define a category whose objects are Hopf algebras and whose morphisms are morphisms of Hopf algebras. We denote this category by

Hopf

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If H is a Hopf algebra, the antipode is antimultiplicative and anticomultiplicative

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

and leaves the unit and counit invariant, i.e.,

$$\lambda_H \circ \eta_H = \eta_H, \ \ \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

A Hopf algebra is cocommutative if

$$\delta_H = c_{H,H} \circ \delta_H.$$

It is easy to see that in this case

$$\lambda_H \circ \lambda_H = id_H.$$

Note that, if $f : H \to D$ is a Hopf algebra morphism the following equality holds:

$$\lambda_D \circ f = f \circ \lambda_H.$$

Hopf trusses

Hopf trusses an related structures in a monoidal setting

- **T. Brzeziński**: Trusses: between braces and rings, Trans. Am. Math. Soc. 372, 4149-4176 (2019). $\mathsf{C} = \mathbb{F}$-Vect

Definition

Let $(H, \varepsilon_H, \delta_H)$ be a coalgebra in C. Assume that there are an algebra structure (H, η_H, μ_H^1), a product $\mu_H^2 : H \otimes H \to H$ and two endomorphism of H denoted by λ_H and σ_H. We will say that

$$(H, \eta_H, \mu_H^1, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H, \sigma_H)$$

is a Hopf truss if:

(i) $H_1 = (H, \eta_H, \mu_H^1, \varepsilon_H, \delta_H, \lambda_H)$ is a Hopf algebra in C.

(ii) $H_2 = (H, \mu_H^2, \varepsilon_H, \delta_H)$ is a non-unital bialgebra in C.

(iii) The morphism σ_H is a coalgebra morphism and the following equality holds:

$$\mu_H^2 \circ (H \otimes \mu_H^1) = \mu_H^1 \circ (\mu_H^2 \otimes \Gamma_{H_1}^{\sigma_H}) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H),$$

where

$$\Gamma_{H_1}^{\sigma_H} = \mu_H^1 \circ ((\lambda_H \circ \sigma_H) \otimes \mu_H^2) \circ (\delta_H \otimes H).$$

Definition

We will say that a Hopf truss is cocommutative if the coalgebra $(H, \varepsilon_H, \delta_H)$ is cocommutative.

Note that, a Hopf truss is a Hopf brace in the sense of I. Angiono, C. Galindo and L. Vendramin

- **I. Angiono, C. Galindo, L. Vendramin**: Hopf braces and Yang-Baxter operators, Proc. Am. Math. Soc. 145, 1981-1995 (2017). $C = \mathbb{F}$-Vect

if σ_H is the identity and there exists a morphism $\lambda_H^2 : H \to H$ such that

$$H_2 = (H, \eta_H, \mu_H^2, \varepsilon_H, \delta_H, \lambda_H^2)$$

is a Hopf algebra.

Notation

Given a Hopf truss, we will denote it by $\mathbb{H} = (H_1, H_2, \sigma_H)$. The morphism σ_H is called the cocycle of $\mathbb{H}$.

The cocycle σ_H of a Hopf truss $\mathbb{H}$ is fully determined by η_H and the product μ_H^2 in the following way:

$$\sigma_H = \mu_H^2 \circ (H \otimes \eta_H).$$

Then, as a consequence of the associativity for the product μ_H^2, we have that

$$\sigma_H \circ \mu_H^2 = \mu_H^2 \circ (H \otimes \sigma_H)$$

holds.

Finally, we know that the pair

$$(H_1, \Gamma_{H_1}^{\sigma_H})$$

is a non-unital left H_2-module algebra.

Definition

Given two Hopf trusses $\mathbb{H}$ and $\mathbb{B}$, a morphism f between the two underlying objects is called a morphism of Hopf trusses if $f : H_1 \to B_1$ is a Hopf algebra morphism and $f : H_2 \to B_2$ is a morphism of non-unital bialgebras.
Then

$$\sigma_B \circ f = f \circ \sigma_H$$

holds.

Hopf trusses together with morphisms of Hopf trusses form a category which we denote by

$$\mathrm{HTr}$$

It is obvious that Hopf braces with morphisms of Hopf braces form a category which we denote by HBr that is a full subcategory of HTr.

Example

A skew truss is a set T with two binary operations $\diamond_1$ and $\diamond_2$ and a map $\omega_T : T \to T$ (called the cocycle) such that the pair $T_1 = (T, \diamond_1)$ is a group with unit $1_{\diamond_1}$, $T_2 = (T, \diamond_2)$ is a semigroup and the following identity

$$a \diamond_2 (b \diamond_1 c) = (a \diamond_2 b) \diamond_1 \omega_T(a)^{\diamond_1} \diamond_1 (a \diamond_2 c)$$

holds for all $a, b, c \in T$. We will denote the previous skew truss by $\mathbb{T} = (T_1, T_2, \omega_T)$. A morphism f between two skew trusses $\mathbb{T} = (T_1, T_2, \omega_T)$ and $\mathbb{S} = (S_1, S_2, \omega_S)$ is a map f between the two underlying sets such that f is a morphism of groups between T_1 and S_1 and of semigroups between T_2 and S_2. With

$$\mathsf{SkTr}$$

we will denote the category of skew trusses.

Then, in Set,

$$\mathsf{SkTr} = \mathsf{HTr}$$

Let $\mathbb{F}$ be a field and C$=\mathbb{F}$-Vect. There exists a functor

$$\mathsf{P}_{skt} : \mathsf{SkTr} \to \mathsf{HTr}$$

given by

$$\mathsf{P}_{skt}(\mathbb{T}) = (\mathbb{F}[T_1], \mathbb{F}[T_2], \sigma_{\mathbb{F}[T]}),$$

where $\sigma_{\mathbb{F}[T]}$ is the linear extension of ω_T and $\lambda_{\mathbb{F}[T]} = (\ )^{\diamond 1}$, on objects and by $\mathsf{P}_{skt}(f) = \mathbb{F}[f]$ on morphisms.

Let $\mathbb{H} = (H_1, H_2, \sigma_H)$ a Hopf truss in $\mathbb{F}$-Vect. There exists a functor

$$\mathsf{R}_{ht} : \mathsf{HTr} \to \mathsf{SkTr}$$

defined by

$$\mathsf{R}_{ht}(\mathbb{H}) = (\mathsf{G}(H_1), \mathsf{G}(H_2), \omega_{\mathsf{G}(H)})$$

on objects and by

$$\mathsf{R}_{ht}(f) = \mathsf{G}(f)$$

on morphisms, where $\omega_{\mathsf{G}(H)}$ the restriction of σ_H to $\mathsf{G}(H)$ and $\mathsf{G}(f)$ the restriction of f to $\mathsf{G}(H)$

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Definition

Let $\mathbb{F}$ be a field and let $\mathbb{H}$ be a Hopf truss in $\mathbb{F}$-Vect. We will say that $\mathbb{H}$ is pointed cosemisimple if the its subjacent coalgebra $(H, \varepsilon_H, \delta_H)$ is pointed and cosemisimple.

Theorem

Let $\mathbb{F}$ be a field and let P_{skt} and R_{ht} be the functors defined in the previous slide. Then,

$$\mathsf{P}_{skt} \dashv \mathsf{R}_{ht}$$

and this adjunction induces an equivalence of categories between SkTr and the full subcategory of HTr of all pointed cosemisimple Hopf trusses in $\mathbb{F}$-Vect.

Hopf trusses and generalized invertible 1-cocycles

Definition

Let $H=(H,\eta_H,\mu_H,\varepsilon_H,\delta_H,\lambda_H)$ be a Hopf algebra in C and let $B=(B,\mu_B,\varepsilon_B,\delta_B)$ be a non-unital bialgebra in C. Assume that H is a non-unital left B-module algebra with action $\phi_H : B\otimes H\rightarrow H$. Let $\pi : B\rightarrow H$ be coalgebra morphism. We will say that π is an generalized invertible 1-cocycle if it is an isomorphism and there exist a coalgebra endomorphism $\theta_\pi : B\rightarrow B$ such that

$$\pi\circ\mu_B=\mu_H\circ((\pi\circ\theta_\pi)\otimes\phi_H)\circ(\delta_B\otimes\pi)$$

holds.

Definition

Let $\pi : B\rightarrow H$ and $\pi' : B'\rightarrow H'$ be generalized invertible 1-cocycles. A morphism between them is a pair (f,g) where $f : B\rightarrow B'$ is a morphism of non-unital bialgebras and $g : H\rightarrow H'$ is a morphism of Hopf algebras satisfying the following identities:

$$f\circ\theta_\pi=\theta_{\pi'}\circ f,\quad g\circ\pi=\pi'\circ f,\quad g\circ\phi_H=\phi_{H'}\circ(f\otimes g).$$

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Definition

Then, with these morphisms, generalized invertible 1-cocycles form a category denoted by GIC. In the following lines an object in GIC will also be denoted by the triple

$$(\pi : B \to H, \theta_\pi).$$

Definition

Note that if $(\pi : B \to H, \theta_\pi)$ is a generalized invertible 1-cocycle such that B is a Hopf algebra, (H, ϕ_H) is a left B-module algebra and $\theta_\pi = id_B$, $(\pi : B \to H, id_B)$ is an invertible 1-cocycle. If we denote the category of invertible 1-cocycles by IC, it is obvious that it is a full subcategory of GIC.

Theorem

The categories GIC and HTr are equivalent.

Proof

Let $\mathbb{H} = (H_1, H_2, \sigma_H)$ be an object in HTr. Then, $(id_H : H_2 \to H_1, \theta_{id_H} = \sigma_H)$ is a generalized invertible 1-cocycle.

On the other hand, let $\mathbb{H} = (H_1, H_2, \sigma_H)$ and $\mathbb{H}' = (H'_1, H'_2, \sigma_{H'})$ be objects in HTr and let $f : \mathbb{H} \to \mathbb{H}'$ be a morphism between them. The pair (f, f) is a morphism in GIC between $(id_H : H_2 \to H_1, \sigma_H)$ and $(id_{H'} : H'_2 \to H'_1, \sigma_{H'})$.

Therefore, there exists a functor

$$\mathrm{E} : \mathrm{HTr} \to \mathrm{GIC}$$

defined on objects by $\mathrm{E}(\mathbb{H}) = (id_H : H_2 \to H_1, \sigma_H)$ and on morphisms by $\mathrm{E}(f) = (f, f)$.

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Let $(\pi : B \to H, \theta_\pi)$ be an object in GIC. Define

$$\mu_H^\pi := \pi \circ \mu_B \circ (\pi^{-1} \otimes \pi^{-1})$$

and $\sigma_\pi := \pi \circ \theta_\pi \circ \pi^{-1}$. Then, $\mathbb{H}_\pi = (H, H_\pi, \sigma_\pi)$, where

$$H_\pi = (H, \mu_H^\pi, \varepsilon_H, \delta_H),$$

is an object in HTr.

Also, if $(f, g) : (\pi : B \to H, \theta_\pi) \to (\pi' : B' \to H', \theta_{\pi'})$ is a morphism in GIC, g is a morphism in HTr between $\mathbb{H}_\pi$ and $\mathbb{H}'_{\pi'}$,

As a consequence of these facts, we have a functor

$$\text{Q} : \text{ GIC} \to \text{HTr}$$

defined by $\text{Q}((\pi : B \to H, \theta_\pi)) = \mathbb{H}_\pi$ on objects and by $\text{Q}((f, g)) = g$ on morphisms.

These functors induce an equivalence between the two categories because, clearly, $\text{QE} = id_{\text{HTr}}$ and $\text{EQ} \simeq \text{Id}_{\text{GIC}}$.

Corollary

The categories IC and HBr are equivalent.

The previous result was proved in:

- **I. Angiono, C. Galindo, L. Vendramin**: Hopf braces and Yang-Baxter operators, Proc. Am. Math. Soc. 145, 1981-1995 (2017). $C = \mathbb{F}$-Vect

Weak Twisted post-Hopf algebras and Hopf trusses

- **Y. Li, Y. Sheng and R. Tang**: Post-Hopf algebras, relative Rota-Baxter operators and solutions of the Yang-Baxter equation, J. Noncommut. Geom 145, 1981-1995 (2024) (in press: DOI 10.4171/JNCG/537).
- **S. Wang**: (Weak) Twisted post-groups, skew trusses and rings arXiv:2307.10535. (2024).

$$C = \mathbb{F}\text{-Vect}$$

Definition

A weak twisted post-Hopf algebra in C is a triple (H, m_H, Φ_H) where H is a Hopf algebra in C and $m_H \colon H \otimes H \to H$ and $\Phi_H \colon H \to H$ are morphisms in C satisfying the following conditions:

(i) m_H is a coalgebra morphism, which means that the following equalities hold:
  (i.1) $\delta_H \circ m_H = (m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$,
  (i.2) $\varepsilon_H \circ m_H = \varepsilon_H \otimes \varepsilon_H$.

(ii) Φ_H is a coalgebra morphism, that is to say:
  (ii.1) $\delta_H \circ \Phi_H = (\Phi_H \otimes \Phi_H) \circ \delta_H$,
  (ii.2) $\varepsilon_H \circ \Phi_H = \varepsilon_H$.

(iii) $\Phi_H \circ \mu_H \circ (\Phi_H \otimes m_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Phi_H \otimes m_H) \circ (\delta_H \otimes \Phi_H)$.

(iv) $m_H \circ (H \otimes m_H) = m_H \circ ((\mu_H \circ (\Phi_H \otimes m_H) \circ (\delta_H \otimes H)) \otimes H)$.

(v) $m_H \circ (H \otimes \mu_H) = \mu_H \circ (m_H \otimes m_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes H \otimes H)$.

The morphism Φ_H will be called the cocycle of the weak twisted post-Hopf algebra H.

Definition

Let (H, m_H, Φ_H) and (B, m_B, Φ_B) be weak twisted post-Hopf algebras in C. We will say that $f\colon (H, m_H, \Phi_H) \to (B, m_B, \Phi_B)$ is a morphism of weak twisted post-Hopf algebras if $f\colon H \to B$ is a Hopf algebra morphism such that

$$f \circ m_H = m_B \circ (f \otimes f), \quad \Phi_B \circ f = f \circ \Phi_H.$$

Therefore, weak twisted post-Hopf algebras give rise to a category that we will denote by

wt-Post-Hopf.

If the underlying Hopf algebra is cocommutative, the structure (H, m_H, Φ_H) is referred to as a cocommutative weak twisted post-Hopf algebra. The corresponding full subcategory is denoted as

coc-wt-Post-Hopf.

Remark

Note that the definition of weak twisted post-Hopf algebras proposed by S. Wang in the category $\mathsf{C} = \mathbb{F}$-Vect, always requires cocommutativity of the underlying Hopf algebra. In the previous definition, this requirement was omitted.

Theorem

Let (H, m_H, Φ_H) be an object in wt-Post-Hopf. If

$(\star)$ $(m_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H) = (m_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$

holds, then $\overline{H} = (H, \overline{\mu}_H, \varepsilon_H, \delta_H)$ is a non-unital bialgebra in C, where

$$\overline{\mu}_H := \mu_H \circ (\Phi_H \otimes m_H) \circ (\delta_H \otimes H).$$

Remark

If C is a symmetric, condition $(\star)$ means that (H, m_H) is in the cocommutativity class of $\overline{H}$ following the notion introduced in:

- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez**: On the (co)-commutativity class of a Hopf algebra and crossed products in a braided category, Comm. Algebra 29, 12, 5857-5878 (2001).

Theorem

Let (H, m_H, Φ_H) be an object in wt-Post-Hopf such that $(\star)$ holds. Then, the triple

$$\overline{\mathbb{H}} = (H, \overline{H}, \Phi_H)$$

is an object in HTr.

As a consequence, if we denote by wt-Post-Hopf* to the full subcategory of wt-Post-Hopf whose objects satisfy $(\star)$, then there exists a functor

$$\mathsf{F}\colon \text{wt-Post-Hopf}^{\star} \longrightarrow \text{HTr}$$

defined on objects by

$$\mathsf{F}((H, m_H, \Phi_H)) = \overline{\mathbb{H}}$$

and on morphisms by the identity.

Hopf trusses an related structures in a monoidal setting

Theorem

Let $\mathbb{H} = (H_1, H_2, \sigma_H)$ be an object in HTr such that the condition

$$(\star)\ \ (\Gamma_{H_1}^{\sigma_H} \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)$$

$$= (\Gamma_{H_1}^{\sigma_H} \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$

holds. Under these hypothesis, $(H_1, \Gamma_{H_1}^{\sigma_H}, \sigma_H)$ is an object in wt-Post-Hopf*.

Remark

When C is symmetric, note that $(\star)$ means that $(H_1, \Gamma_{H_1}^{\sigma_H})$ is in the cocommutativity class of H_2.

From now on, let's denote by $\mathsf{HTr}^{\star}$ to the full subcategory of HTr whose objects satisfy condition $(\star)$. Therefore, there exists a functor

$$\mathsf{G}\colon \mathsf{HTr}^{\star} \longrightarrow \text{wt-Post-Hopf}^{\star}$$

acting on objects by $\mathsf{G}(\mathbb{H}) = (H_1, \Gamma_{H_1}^{\sigma_H}, \sigma_H)$ and on morphisms by the identity.

Remark

Note also that if (H, m_H, Φ_H) is an object in wt-Post-Hopf*, the Hopf truss $\mathsf{F}((H, m_H, \Phi_H)) = \overline{\mathbb{H}}$ belongs to the category $\mathsf{HTr}^{\star}$, so F admits a factorization from wt-Post-Hopf* to $\mathsf{HTr}^{\star}$.

Theorem

The categories wt-Post-Hopf* and $\mathsf{HTr}^{\star}$ are isomorphic.

Corollary

Categories coc-wt-Post-Hopf and coc-HTr are isomorphic.

Weak twisted Relative Rota-Baxter operators and Hopf trusses

- **M. Goncharov**: Rota-Baxter operators on cocommutative Hopf algebras, J. Algebra 582, 39-56 (2021).
- **Y. Li, Y. Sheng and R. Tang**: Post-Hopf algebras, relative Rota-Baxter operators and solutions of the Yang-Baxter equation, J. Noncommut. Geom 145, 1981-1995 (2024) (in press: DOI 10.4171/JNCG/537).

$C = \mathbb{F}$-Vect

Hopf trusses an related structures in a monoidal setting

Definition

Let $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ be a Hopf algebra and let $B = (B, \mu_B, \varepsilon_B, \delta_B)$ be a non-unital bialgebra in C. Suppose that there exists a morphism $\varphi_H : B \otimes H \to H$ such that (H, φ_H) is a non-unital left B-module algebra-coalgebra. We will say that a coalgebra morphism

$$T\colon H \to B$$

is a weak twisted relative Rota-Baxter operator if there exists $\Psi_H\colon H \to H$ a coalgebra morphism, called the cocycle of T, such that the following conditions hold:

(i) $\mu_B \circ (T \otimes T) = T \circ \mu_H \circ (\Psi_H \otimes (\varphi_H \circ (T \otimes H))) \circ (\delta_H \otimes H)$,

(ii) $\Psi_H \circ \mu_H \circ (\Psi_H \otimes (\varphi_H \circ (T \otimes H))) \circ (\delta_H \otimes H) = \mu_H \circ (\Psi_H \otimes (\varphi_H \circ (T \otimes H))) \circ (\delta_H \otimes \Psi_H)$.

In what follows we will denote weak twisted relative Rota-Baxter operators by

$$\left(T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H, \Psi_H\right).$$

If we define $\mathfrak{m}_H$ by

$$\mathfrak{m}_H = \varphi_H \circ (T \otimes H) : H \otimes H \to H,$$

conditions (i) and (ii) of previous definition are equivalent to

(i) $\mu_B \circ (T \otimes T) = T \circ \mu_H \circ (\Psi_H \otimes \mathfrak{m}_H) \circ (\delta_H \otimes H)$,

(ii) $\Psi_H \circ \mu_H \circ (\Psi_H \otimes \mathfrak{m}_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Psi_H \otimes \mathfrak{m}_H) \circ (\delta_H \otimes \Psi_H)$.

Definition

Let $(T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H, \Psi_H)$ and $(T' \begin{array}{c} H' \\ \downarrow \\ B' \end{array}, \varphi_{H'}, \Psi_{H'})$ be weak twisted relative Rota-Baxter operators. We will say that

$$(f,g)\colon (T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H, \Psi_H) \to (T' \begin{array}{c} H' \\ \downarrow \\ B' \end{array}, \varphi_{H'}, \Psi_{H'}),$$

where $f\colon H \to H'$ is a Hopf algebra morphism and $g\colon B \to B'$ is a morphism of non-unital bialgebras, is a morphism of weak twisted relative Rota-Baxter operators if the following conditions hold:

$$T' \circ f = g \circ T, \quad f \circ \Psi_H = \Psi_{H'} \circ f, \quad f \circ \varphi_H = \varphi_{H'} \circ (g \otimes f).$$

So, weak twisted relative Rota-Baxter operators give rise to a category that we will denote by wtr-RB.

Remark

Consider $(T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \;, \varphi_H, \Psi_H)$ a weak twisted relative Rota-Baxter operator. Due to being T a coalgebra morphism and (H, φ_H) a non-unital left B-module algebra-coalgebra, it is straightforward to prove that the following equalities hold:

$$\mathfrak{m}_H \circ (H \otimes \eta_H) = \varepsilon_H \otimes \eta_H,$$

$$\mathfrak{m}_H \circ (H \otimes \mu_H) = \mu_H \circ (\mathfrak{m}_H \otimes \mathfrak{m}_H) \circ \delta_{H \otimes H},$$

$$\varepsilon_H \circ \mathfrak{m}_H = \varepsilon_H \otimes \varepsilon_H,$$

$$\delta_H \circ \mathfrak{m}_H = (\mathfrak{m}_H \otimes \mathfrak{m}_H) \circ \delta_{H \otimes H}.$$

Moreover, the equality

$$\mathfrak{m}_H \circ ((\mu_H \circ (\Psi_H \otimes \mathfrak{m}_H) \circ (\delta_H \otimes H)) \otimes H) = \mathfrak{m}_H \circ (H \otimes \mathfrak{m}_H)$$

also holds.

Remark

Note that if (f, g) is a morphism between the weak twisted relative Rota-Baxter operators $(T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H, \Psi_H)$ and $(T' \begin{array}{c} H' \\ \downarrow \\ B' \end{array}, \varphi_{H'}, \Psi_{H'})$, then

$$f \circ \mathfrak{m}_H = \mathfrak{m}_{H'} \circ (f \otimes f)$$

holds.

Hopf trusses an related structures in a monoidal setting

Theorem

Let $(T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \;, \varphi_H, \Psi_H)$ be a weak twisted relative Rota-Baxter operator such that

$$(\star) \;\; (\mathfrak{m}_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((c_{H,H} \circ \delta_H) \otimes H)$$
$$= (\mathfrak{m}_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$$

holds. Then, $\widetilde{H} = (H, \widetilde{\mu}_H, \varepsilon_H, \delta_H)$, where

$$\widetilde{\mu}_H := \mu_H \circ (\Psi_H \otimes \mathfrak{m}_H) \circ (\delta_H \otimes H),$$

is a non-unital bialgebra in C.

Remark

Note that, $(H, \mathfrak{m}_H)$ is a non-unital left $\widetilde{H}$-module. Then, if C is symmetric, we can say that $(H, \mathfrak{m}_H)$ is in the cocommutativity class of $\widetilde{H}$ because $(\star)$ holds.

Theorem

Let $(T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \; , \varphi_H, \Psi_H)$ be a weak twisted relative Rota-Baxter operator such that $(\star)$ holds. Then, the triple $\widetilde{\mathbb{H}} = (H, \widetilde{H}, \Psi_H)$ is an object in $\text{HTr}^{\star}$.

Remark

Then, if we denote by wtr-RB* to the full subcategory of wtr-RB of objects satisfying the condition $(\star)$, there exists a functor

$$\Omega \colon \text{wtr-RB}^{\star} \longrightarrow \text{HTr}^{\star}$$

defined on objects by

$$\Omega((T \; \begin{matrix} H \\ \downarrow \\ B \end{matrix} \; , \varphi_H, \Psi_H)) = \widetilde{\mathbb{H}}$$

and on morphisms by $\Omega((f, g)) = f$.

Theorem

Let $\mathbb{H} = (H_1, H_2, \sigma_H)$ be an object in $\mathsf{HTr}^\star$. Then, the triple

$$\left(id_H \begin{array}{c} H_1 \\ \downarrow \\ H_2 \end{array}, \Gamma_{H_1}^{\sigma_H}, \sigma_H\right)$$

is a weak twisted relative Rota-Baxter operator satisfying condition $(\star)$.

Remark

Thus, from the above theorem, it follows that there exists a functor

$$\Lambda \colon \mathsf{HTr}^\star \longrightarrow \mathsf{wtr\text{-}RB}^\star$$

acting on objects by

$$\Lambda(\mathbb{H}) = \left(id_H \begin{array}{c} H_1 \\ \downarrow \\ H_2 \end{array}, \Gamma_{H_1}^{\sigma_H}, \sigma_H\right)$$

and on morphisms by $\Lambda(f) = (f, f)$.

Hopf trusses an related structures in a monoidal setting

Theorem

The functor Λ is left adjoint to the functor Ω.

Remark

Consider the full subcategory of wtr-RB* consisting of all weak twisted relative Rota-Baxter operators $(T \begin{array}{c} H \\ \downarrow \\ B \end{array}, \varphi_H, \Psi_H)$, such that T is an isomorphism in C. We will denote this subcategory by

$$\text{wtr-RB}^{\star}_{\text{iso}}.$$

Moreover, take into account that the image of the functor Λ are in this subcategory we have a functor

$$\Lambda \colon \text{HTr}^{\star} \longrightarrow \text{wtr-RB}^{\star}_{\text{iso}}.$$

Thus, if we denote by Ω' the restriction of functor Ω to the subcategory wtr-RB$^{\star}_{\text{iso}}$, the following result states that Λ and Ω' give rise to a categorical equivalence between wtr-RB$^{\star}_{\text{iso}}$ and HTr*.

Theorem

The categories $\mathrm{HTr}^{\star}$ and $\text{wtr-RB}^{\star}_{\text{iso}}$ are equivalent.

Corollary

The categories $\mathrm{HTr}^{\star}$, $\text{wtr-RB}^{\star}_{\text{iso}}$ and $\text{wt-Post-Hopf}^{\star}$ are equivalent.

Corollary

The categories coc-HTr and $\text{coc-wtr-RB}_{\text{iso}}$ are equivalent.

Corollary

The categories coc-HTr, $\text{coc-wtr-RB}_{\text{iso}}$ and coc-wt-Post-Hopf are equivalent.

Modules for Hopf trusses

Hopf trusses an related structures in a monoidal setting

Definition

Let $\mathbb{H}$ be a Hopf truss. A left $\mathbb{H}$-module is a triple (M, ψ_M^1, ψ_M^2), where (M, ψ_M^1) is a left H_1-module, (M, ψ_M^2) is a non-unital left H_2-module and the following identity

$$\psi_M^2 \circ (H \otimes \psi_M^1) = \psi_M^1 \circ (\mu_H^2 \otimes \Gamma_M^{\sigma_H}) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes H \otimes M)$$

holds, where

$$\Gamma_M^{\sigma_H} = \psi_M^1 \circ ((\lambda_H \circ \sigma_H) \otimes \psi_M^2) \circ (\delta_H \otimes M).$$

Given two left $\mathbb{H}$-modules (M, ψ_M^1, ψ_M^2) and (N, ψ_N^1, ψ_N^2), a morphism $f : M \to N$ is called a morphism of left $\mathbb{H}$-modules if f is a morphism of left H_1-modules and left non-unital H_2-modules. Left $\mathbb{H}$-modules with morphisms of left $\mathbb{H}$-modules form a category which we denote by

$${}_{\mathbb{H}}\mathrm{Mod}.$$

Examples

(i) Let $\mathbb{H}$ be a Hopf truss. The triple $(H, \psi_H^1 = \mu_H^1, \psi_H^2 = \mu_H^2)$ is an example of left $\mathbb{H}$-module. Also, if K is the unit object of C, $(K, \psi_K^1 = \varepsilon_H, \psi_K^2 = \varepsilon_H)$ is a left $\mathbb{H}$-module called the trivial module.

(ii) If X is an object in C, $\mathbb{H} \otimes X = (H \otimes X, \psi_{H\otimes X}^1 = \mu_H^1 \otimes X, \psi_{H\otimes X}^2 = \mu_H^2 \otimes X)$ is an example of left $\mathbb{H}$-module. Also, if $f : X \to X'$ is a morphism in C, $H \otimes f$ is a morphism in ${}_{\mathbb{H}}$Mod between $\mathbb{H} \otimes X$ and $\mathbb{H} \otimes X'$. Therefore, there exist a functor, called the induction functor, $\mathbb{H} \otimes - : \mathsf{C} \to {}_{\mathbb{H}}\mathsf{Mod}$ defined on objects by

$$\mathbb{H} \otimes -(X) = \mathbb{H} \otimes X$$

and on morphisms by $\mathbb{H} \otimes -(f) = \mathbb{H} \otimes f$.

Remark

If the a Hopf truss $\mathbb{H}$ is a Hopf brace and we assume that a (M, ψ_M^2) is a left H_2-module, we obtain the definition of module over a Hopf brace introduced in

- **R. González Rodríguez**: The fundamental theorem of Hopf modules for Hopf braces, Linear Multilinear Algebra 70, 5146-5156 (2022).

Definition

Let $(\pi : B \to H, \theta_\pi)$ be a generalized invertible 1-cocycle. A left module over

$$(\pi : B \to H, \theta_\pi)$$

is a 6-tuple $(M, N, \phi_M, \varphi_M, \phi_N, \gamma)$ where:

(i) $\phi_M : B \otimes M \to M$ is a morphism in C.

(ii) (M, φ_M) is a left H-module.

(iii) (N, ϕ_N) is a non-unitary left B-module.

(iv) The equality

$$\phi_M \circ (B \otimes \varphi_M) = \varphi_M \circ (\phi_H \otimes \phi_M) \circ (B \otimes c_{A,H} \otimes M) \circ (\delta_B \otimes H \otimes M).$$

holds.

(v) $\gamma : N \to M$ is an isomorphism in C such that

$$\gamma \circ \phi_N = \varphi_M \circ ((\pi \circ \theta_\pi) \otimes \phi_M) \circ (\delta_B \otimes \gamma).$$

Definition

Let $(M, N, \phi_M, \varphi_M, \phi_N, \gamma)$ and $(M', N', \phi_{M'}, \varphi_{M'}, \phi_{N'}, \gamma')$ be left modules over a generalized invertible 1-cocycle $(\pi : B \to H, \theta_\pi)$. A morphism between them is a pair (h, l) of morphisms in C such that:

(i) The morphism $h : M \to M'$ satisfies $h \circ \phi_M = \phi_{M'} \circ (B \otimes h)$ and is left H-linear.

(ii) The morphism $l : N \to N'$ is left B-linear.

(iii) The following identity holds:

$$h \circ \gamma = \gamma' \circ l.$$

With the obvious composition of morphisms, left modules over a generalized invertible 1-cocycle $(\pi : B \to H, \theta_\pi)$ with action ϕ_H form a category that we will denote by

$$_{(\pi, \phi_H, \theta_\pi)}\text{Mod}$$

If $(\pi : B \to H, \theta_\pi)$ is a generalized invertible 1-cocycle, the 6-tuple $(H, B, \phi_H, \mu_H, \mu_B, \pi)$ is an example of left module over $(\pi : B \to H, \theta_\pi)$.

Theorem

Let (f, g) be a morphism between the generalized invertible 1-cocycles $(\pi : B \to H, \theta_\pi)$ and $(\pi' : B' \to H', \theta_{\pi'})$. Then, there exists a functor

$$\mathsf{M}_{(f,g)} : {}_{(\pi',\phi_{H'},\theta_{\pi'})}\mathsf{Mod} \to {}_{(\pi,\phi_H,\theta_\pi)}\mathsf{Mod}$$

defined on objects by

$$\mathsf{M}_{(f,g)}((P, Q, \phi_P, \varphi_P, \phi_Q, \tau))$$
$$= (P, Q, \phi_P^\pi = \phi_P \circ (f \otimes P), \varphi_P^\pi = \varphi_P \circ (g \otimes P), \phi_Q^\pi = \phi_Q \circ (f \otimes Q), \tau)$$

and on morphisms by the identity.

For all generalized invertible 1-cocycle $(\pi : B \to H, \theta_\pi)$, (π, id_H) is an isomorphism between the generalized invertible 1-cocycles $(\pi : B \to H, \theta_\pi)$ and $(id_H : H_\pi \to H, \sigma_\pi)$ Therefore, the functor

$$\mathsf{M}_{(\pi,id_H)} : {}_{(id_H,\Gamma_H^{\sigma_\pi},\sigma_\pi)}\mathsf{Mod} \to {}_{(\pi,\phi_H,\theta_\pi)}\mathsf{Mod}$$

is an isomorphism where $\mathsf{M}_{(\pi^{-1},id_H)} : {}_{(\pi,\phi_H,\theta_\pi)}\mathsf{Mod} \to {}_{(id_H,\Gamma_H^{\sigma_\pi},\sigma_\pi)}\mathsf{Mod}$ is the inverse.

Theorem

Let $\mathbb{H}$ be a Hopf truss. There exists a functor

$$\mathsf{G}_{\mathbb{H}} : \ {}_{\mathbb{H}}\mathsf{Mod} \ \rightarrow \ {}_{(id_H, \Gamma_{H_1}^{\sigma_H}, \sigma_H)}\mathsf{Mod}$$

defined on objects by

$$\mathsf{G}_{\mathbb{H}}((M, \psi_M^1, \psi_M^2)) = (M, M, \widehat{\phi}_M = \Gamma_M^{\sigma_H}, \widehat{\varphi}_M = \psi_M^1, \overline{\phi}_M = \psi_M^2, id_M)$$

and on morphisms by $\mathsf{G}_{\mathbb{H}}(f) = (f, f)$.

Hopf trusses an related structures in a monoidal setting

Theorem

Let $(\pi : B \to H, \theta_\pi)$ be a generalized invertible 1-cocycle. There exists a functor

$$\mathsf{H}^\pi : {}_{(\pi,\phi_H,\theta_\pi)}\mathsf{Mod} \to {}_{\mathbb{H}_\pi}\mathsf{Mod}$$

defined on objects by

$$\mathsf{H}^\pi((M, N, \phi_M, \varphi_M, \phi_N, \gamma)) = (M, \overline{\psi}_M^1 = \varphi_M, \overline{\psi}_M^2 = \gamma \circ \phi_N \circ (\pi^{-1} \otimes \gamma^{-1}))$$

and on morphisms by $\mathsf{H}^\pi((h, l)) = h$.

Theorem

Let $(\pi : B \to H, \theta_\pi)$ be a generalized invertible 1-cocycle. Then the categories ${}_{(\pi,\phi_H,\theta_\pi)}\mathsf{Mod}$ and ${}_{\mathbb{H}_\pi}\mathsf{Mod}$ are equivalent.

Proof

$$\mathsf{H}^\pi \circ (\mathsf{M}_{(\pi, id_H)} \circ \mathsf{G}_{\mathbb{H}_\pi}) = \mathrm{id}_{\ {}_{\mathbb{H}_\pi}\mathsf{Mod}},$$

$$(\mathsf{M}_{(\pi, id_H)} \circ \mathsf{G}_{\mathbb{H}_\pi}) \circ \mathsf{H}^\pi \backsimeq \mathrm{id}_{\ {}_{(\pi,\phi_H,\theta_\pi)}\mathsf{Mod}}.$$

When we particularize the previous results to modules asociated to Hopf braces and invertible 1-cocycles we have the categorical equivalences obtained in:

- **J.M. Fernández Vilaboa, R. González Rodríguez, B. Ramos Pérez, A.B. Rodríguez Raposo**: Modules over invertible 1-cocycles, Turkish J. Math. 70, 5146-5156 (2024).

The Fundamental Theorem of Hopf modules for Hopf trusses

Hopf trusses an related structures in a monoidal setting

In this section we will assume that C admits equalizers. As a consequence every idempotent morphism in C splits, i.e., if $q : M \to M$ is a morphism in C such that $q = q \circ q$, there exists an object $I(q)$, called the image of q, and morphisms $i : I(q) \to M$ and $p : M \to I(q)$ such that $q = i \circ p$ and $p \circ i = id_{I(q)}$. The morphisms p and i will be called a factorization of q. Note that $I(q)$, p and i are unique up to isomorphism.

Definition

Let D be a coalgebra in C. The pair (M, ρ_M) is a left D-comodule if M is an object in C and $\rho_M : M \to D \otimes M$ is a morphism in C satisfying

$$(\varepsilon_D \otimes M) \circ \rho_M = id_M, \quad (D \otimes \rho_M) \circ \rho_M = (\delta \otimes M) \circ \rho_M.$$

Given two left D-comodules (M, ρ_M) and (N, ρ_N), a morphism $f : M \to N$ in C is a morphism of left D-comodules if $(D \otimes f) \circ \rho_M = \rho_N \circ f$. Left D-comodules with morphisms of left D-comodules form a category which we denote by ^{D}Comod.

Hopf trusses an related structures in a monoidal setting

Definition

Let D be a coalgebra such that there exits a coalgebra morphism $e : K \rightarrow D$. Let (M, ρ_M) be a left D-comodule. We define the subobject of coinvariants of M, denoted by M_e^{coD}, as the equalizer object of ρ_M and $e \otimes M$. Then, we have an equalizer diagram

$$M_e^{coD} \xrightarrow{\quad j_M^e \quad} M \underset{e \otimes M}{\overset{\rho_M}{\rightrightarrows}} D \otimes M$$

where j_M^e denotes the equalizer (inclusion) morphism.

Notation

If H is a Hopf algebra, the unit η_H is a coalgebra morphism. Then, Let (M, ρ_M) be a left D-comodule, we will denote the equalizer object of ρ_M and $\eta_H \otimes M$ by M^{coH} and the equalizer morphism by j_M.

Definition

Let B be a non-unital bialgebra. A non-unital left B-Hopf module is a triple (M, φ_M, ρ_M) where (M, φ_M) is a non-unital left B-module, (M, ρ_M) is a left B-comodule and

$$\varphi_M \circ \rho_M = (\mu_B \otimes \varphi_M) \circ (B \otimes c_{B,B} \otimes M) \circ (\delta_B \otimes \rho_M)$$

holds. Non-unital left B-Hopf modules with left linear and colinear morphisms form a category which we denote by B-Hopf-mod.

Remark

The definition for left H-Hopf modules over a Hopf algebra H is similar changing non-unital left H-modules by left H-modules. Then, in this case we will denote the category of H-Hopf modules by H-Hopf-Mod.

Hopf trusses an related structures in a monoidal setting

Let H be a Hopf algebra, it easy to show that, if (M, φ_M, ρ_M) is a left H-Hopf module, the endomorphism $q_M : M \to M$, defined by

$$q_M = \varphi_M \circ (\lambda_H \otimes M) \circ \rho_M$$

is idempotent and satisfies $\rho_M \circ q_M = \eta_H \otimes q_M$. Therefore, there exists a unique morphism

$$t_M : M \to M^{coH}$$

such that

$$t_M \circ j_M = q_M.$$

Let $I(q_M)$ be the image of the idempotent morphism q_M and let $i_M : I(q_M) \to M$ and $p_M : M \to I(q_M)$ be the morphisms such that that $q_M = i_M \circ p_M$ and $p_M \circ i_M = id_{I(q_M)}$. The morphism

$$\omega_M = t_M \circ i_M : I(q_M) \to M^{coH}$$

is an isomorphism with inverse $\omega_M^{-1} = p_M \circ j_M$.

The object $H \otimes M^{coH}$ is a left H-Hopf module with action

$$\varphi_{H\otimes M^{coH}} = \mu_H \otimes M^{coH}$$

and coaction

$$\rho_{H\otimes M^{coH}} = \delta_H \otimes M^{coH}.$$

The Fundamental Theorem of Hopf modules asserts that $H \otimes M^{coH}$ is isomorphic to M in the category H-Hopf-Mod. The isomorphism is defined by

$$\theta_M = \varphi_M \circ (H \otimes j_M) : H \otimes M^{coH} \to M$$

where $\theta_M^{-1} = (H \otimes t_M) \circ \rho_M$.

In the same way as in the case of M^{coH}, if X is an object in C, the tensor product $H \otimes X$, with the action and coaction induced by the product and the coproduct of H, is a left H-Hopf module. Then, there exists a functor

$$\mathsf{F} = H \otimes - : \mathsf{C} \ \to \text{H-Hopf-Mod}$$

called the induction functor.

Also, for all $M \in$ H-Hopf-Mod, the construction of M^{coH} is functorial. Thus, there exists a new functor

$$\mathsf{G} = (\ \ )^{coH} : \text{H-Hopf-Mod} \ \to \mathsf{C},$$

called the functor of coinvariants, such that $\mathsf{F} \dashv \mathsf{G}$.

Moreover, F and G induce an equivalence between the categories H-Hopf-Mod and C.

Definition

Let $\mathbb{H} = (H_1, H_2, \sigma_H)$ be a Hopf truss. A left Hopf module over $\mathbb{H}$ (left $\mathbb{H}$-Hopf module) is a 4-tuple $(M, \psi_M^1, \psi_M^2, \rho_M)$ such that:

(i) The triple (M, ψ_M^1, ψ_M^2) is a left $\mathbb{H}$-module.

(ii) The triple (M, ψ_M^1, ρ_M) is a left H_1-Hopf module.

(iii) The triple (M, ψ_M^2, ρ_M) is a non-unital left H_2-Hopf module.

(iv) The identity $\psi_M^1 \circ (\sigma_H \otimes j_M) = \psi_M^2 \circ (H \otimes j_M)$ holds.

A morphism of left Hopf modules over $\mathbb{H}$ is a morphism of left $\mathbb{H}$-modules and left H-comodules. Left Hopf modules over $\mathbb{H}$ with morphisms of left Hopf modules form a category which we denote by $\mathbb{H}$-Hopf-Mod.

Note that, this definition is a generalization to the Hopf truss setting of the notion of Hopf module over a Hopf brace introduced in

- **R. González Rodríguez**, The fundamental theorem of Hopf modules for Hopf braces, Linear Multilinear Algebra 70, 5146-5156 (2022).

Example

Let X be an object in C and let $\mathbb{H} = (H_1, H_2, \sigma_H)$ be a Hopf truss. Then, the 4-tuple

$$(H \otimes X, \psi^1_{H\otimes X} = \mu^1_H \otimes X, \psi^2_{H\otimes X} = \mu^2_H \otimes X, \rho_{H\otimes X} = \delta_H \otimes X)$$

is a left $\mathbb{H}$-Hopf module.

Theorem

Let $\mathbb{H}$ be a Hopf truss. There exists a functor $\mathsf{V} = H \otimes - : \mathsf{C} \rightarrow \mathbb{H}$-Hopf, called the induction functor, defined on objects by

$$\mathsf{V}(X) = (H \otimes X, \psi^1_{H\otimes X}, \psi^2_{H\otimes X}, \rho_{H\otimes X})$$

and on morphisms by $\mathsf{V}(f) = H \otimes f$.

Theorem (Fundamental Theorem of Hopf modules)

Let $\mathbb{H}$ be a Hopf truss and let $(M, \varphi_M, \psi_M, \rho_M)$ be an object in $\mathbb{H}$-Hopf-Mod. Then $(M, \psi^1_M, \psi^2_M, \rho_M)$ and $\mathsf{V}(M^{coH_1})$ are isomorphic in $\mathbb{H}$-Hopf-Mod.

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Theorem

Let $\mathbb{H}$ be a Hopf truss. There exists a functor $\mathrm{W} = (\ \ )^{co\mathbb{H}} : \mathbb{H}\text{-Hopf-Mod} \to \mathsf{C}$, called the functor of coinvariants, defined on objects by

$$\mathrm{W}((M, \psi_M^1, \psi_M^2, \rho_M)) = M^{co\mathbb{H}},$$

where $M^{co\mathbb{H}} = M^{coH_1}$ and on morphisms $f : M \to N$ by $\mathrm{W}(f) = f^{co\mathbb{H}}$, where $f^{co\mathbb{H}} = f^{coH_1}$ is the unique morphism such that $j_N \circ f^{coH_1} = f \circ j_M$.

Theorem

Let $\mathbb{H}$ a Hopf truss. The induction functor $\mathrm{V} = H \otimes - : \mathsf{C} \to \mathbb{H}\text{-Hopf-Mod}$ is left adjoint of the functor of coinvariants $\mathrm{W} = (\ \ )^{co\mathbb{H}} : \mathbb{H}\text{-Hopf-Mod} \to \mathsf{C}$ and they induce a categorical equivalence between $\mathbb{H}$-Hopf-Mod and C.

If we particularize the previous theorems to the case of Hopf braces we have the Fundamental Theorem of Hopf Modules and the associated categorical equivalence obtained in

- **R. González Rodríguez**: The fundamental theorem of Hopf modules for Hopf braces, Linear Multilinear Algebra 70, 5146-5156 (2022).

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Thank you

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