Factorizations and double cross products of Hopf quasigroups

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Hopf quasigroups

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Recall that a monoidal category is a category C together with a functor

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$$

called tensor product, an object K of C, called the unit object, and families of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in C, called associativity, right unit and left unit constraints, respectively, satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

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where for each object X in C, id_X denotes the identity morphism of X.

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A monoidal category is called strict if the constraints of the previous paragraph are identities.

It is a well-known fact that every non-strict monoidal category is monoidal equivalent to a strict one. This lets us to treat monoidal categories as if they were strict and, as a consequence, the results proved in a strict setting hold for every non-strict monoidal category.

A braiding for a strict monoidal category C is a natural family of isomorphisms

$$c_{M,N} : M \otimes N \to N \otimes M$$

subject to the conditions

$$c_{M,N\otimes P} = (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P), \quad c_{M\otimes N,P} = (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P})$$

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If

$$c_{N,M} \circ c_{M,N} = id_{M\otimes N}$$

for all M, N in C, we will say that C is symmetric.

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Notation

For simplicity of notation, given objects M, N, P in C and a morphism $f : M \to N$, we will write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

Factorizations and double cross products of Hopf quasigroups

Definition

A magma in $\mathcal{C}$ is a pair $A = (A, \mu_A)$, where A is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ (product) is a morphism in $\mathcal{C}$.
A unital magma in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$, where (A, μ_A) is a magma in $\mathcal{C}$ and $\eta_A : K \to A$ (unit) is a morphism in $\mathcal{C}$ such that

$$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A).$$

A monoid in $\mathcal{C}$ is a unital magma $A = (A, \eta_A, \mu_A)$ in $\mathcal{C}$ satisfying

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Definition

Given two unital magmas (monoids) A and B, $f : A \to B$ is a morphism of unital magmas (monoids) if $f \circ \eta_A = \eta_B$ and $\mu_B \circ (f \otimes f) = f \circ \mu_A$.

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If A, B are are unital magmas (monoids), $A \otimes B$ is also a unital magma (monoid) where

$$\eta_{A\otimes B} = \eta_A \otimes \eta_B, \quad \mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B).$$

Definition

A comagma in $\mathcal{C}$ is a pair $D = (D, \delta_D)$, where D is an object in $\mathcal{C}$ and $\delta_D : D \to D \otimes D$ (coproduct) is a morphism in $\mathcal{C}$. A counital comagma in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where (D, δ_D) is a comagma in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit) is a morphism in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. A comonoid in $\mathcal{C}$ is a counital comagma in $\mathcal{C}$ satisfying $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, i.e., the coproduct δ_D is coassociative.

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If D and E are counital comagmas (comonoids) in $\mathcal{C}$, $f : D \to E$ is a morphism of counital comagmas (comonoids) if $\varepsilon_E \circ f = \varepsilon_D$, and $(f \otimes f) \circ \delta_D = \delta_E \circ f$.

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Moreover, if D, E are counital comagmas (comonoids), $D \otimes E$ is a counital comagma (comonoid), where

$$\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E, \quad \delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E).$$

Definition

Let $f : D \to A$ and $g : D \to A$ be morphisms between a comagma D and a magma A. We define the convolution product by

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_D.$$

Factorizations and double cross products of Hopf quasigroups

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If A is unital and D counital, we will say that f is convolution invertible if there exists $f^* : D \to A$ such that

$$f * f^* = f^* * f = \varepsilon_D \otimes \eta_A.$$

Definition

A non-associative bimonoid in the category $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that ε_H and δ_H are morphisms of unital magmas (equivalently, η_H and μ_H are morphisms of counital comagmas). Then the following identities hold:

$$\varepsilon_H \circ \eta_H = id_K, \quad \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H,$$

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Definition

A non-associative bimonoid is called cocommutative if $\delta_H = c_{H,H} \circ \delta_H$.

Definition

J. Klim, S. Majid: Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323 (2010), 3067-3110. ($\mathcal{C} = \mathbb{F}$-Vect)

A Hopf quasigroup H in $\mathcal{C}$ is a non-associative bimonoid such that there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)$$

and

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H).$$

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and

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Note that composing with $H \otimes \eta_H$ in the first equality we obtain that

$$\lambda_H * id_H = \varepsilon_H \otimes \eta_H,$$

and composing with $\eta_H \otimes H$ in the second one we obtain

$$id_H * \lambda_H = \varepsilon_H \otimes \eta_H.$$

Therefore, λ_H is convolution invertible and $\lambda_H^* = id_H$.

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Also, the antipode is antimultiplicative and anticomultiplicative, i.e.,

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$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$$

and leaves the unit and the counit invariable:

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Then the equality

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holds.

A Hopf quasigroup H is associative if, and only if, H is a Hopf monoid (algebra).

Example

Suppose that $\mathcal{C} = \text{Set}$. Then L is a Hopf quasigroup in $\mathcal{C}$ if, and only if, L is an IP loop. An IP loop is a set L with a product, identity e_L, and with the property that for each $u \in L$ there exists $u^{-1} \in L$ (the inverse of u) such that

$$u^{-1}(uv) = v, \quad (vu)u^{-1} = v, \quad \forall \; v \in L.$$

As a consequence, it is easy to show that , if L is an IP loop, for all $u \in L$ the element $u^{-1}u = e_L = uu^{-1}$, u^{-1} is unique and $(u^{-1})^{-1} = u$. Moreover, $(uv)^{-1} = v^{-1}u^{-1}$ holds for any pair of elements $u, v \in L$.
Note that in this case L is a cocommutative Hopf quasigroup because

$$\delta_L(u) = (u, u).$$

Example

Suppose that $\mathcal{C}$ is $\mathbb{F}$-Vect. Let L be an IP loop. Then, the loop algebra

$$\mathbb{F}[L] = \bigoplus_{u \in L} \mathbb{F}u$$

is a cocommutative non-associative bimonoid with unit $\eta_{\mathbb{F}[L]}(1_{\mathbb{F}}) = e_L$, product defined by linear extension of the one defined in L, and coproduct and counit

$$\delta_{\mathbb{F}[L]}(u) = u \otimes u, \quad \varepsilon_{\mathbb{F}[L]}(u) = 1_{\mathbb{F}}.$$

Also, it is a Hopf quasigroup where the antipode is defined by the linear extension of the map

$$\lambda_{\mathbb{F}[L]}(u) = u^{-1}.$$

Example

Let $\mathbb{F}$ be a field. A Malcev algebra over $\mathbb{F}$ is an anticommutative algebra $(M, [,])$ such that

$$[J(a,b,c),a] = J(a,b,[a,c]),$$

where $J(a,b,c) = [[a,b],c] - [[a,c],b] - [a,[b,c]]$ denotes the Jacobian in a, b, c. As was proved in

> **J. Klim, S. Majid**: Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323 (2010), 3067-3110,

if the characteristic of $\mathbb{F}$ is different of 2 and 3, then the universal enveloping algebra $U(M)$, introduced by

> **J.M. Pérez Izquierdo, I.P. Shestakov**: An envelope for Malcev algebras, J. Algebra 272 (2004), 379-393,

admits a cocommutative Hopf quasigroup structure.

Factorizations and double cross products of Hopf quasigroups

Distributive laws for Hopf quasigroups

Definition

Let H, A be Hopf quasigroups. A morphism

$$\Psi : H \otimes A \to A \otimes H$$

is said to be a distributive law of H over A if the following identities

$$\Psi \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A) \circ (\lambda_H \otimes \lambda_A \otimes A),$$

$$\Psi \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = (A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi) \circ (H \otimes \lambda_H \otimes \lambda_A),$$

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$$\Psi \circ (H \otimes \eta_A) = \eta_A \otimes H, \quad \Psi \circ (\eta_H \otimes A) = A \otimes \eta_H,$$

hold.

If the antipodes of H and A are isomorphisms, the two first identities are equivalent to

$$\Psi \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A),$$

$$\Psi \circ (\mu_H \otimes A) = (A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi),$$

respectively. Then, in this case, the conditions of the definition of distributive law for Hopf quasigroups are the ones that we can find in the classical definition of distributive law between monoids, i.e., Ψ is compatible with the unit and the product of A and H.

Definition

Let H, A be Hopf quasigroups and let $\Psi : H \otimes A \to A \otimes H$ be a distributive law of H over A. The distributive law Ψ is said to be comonoidal if it is a comonoid morphism, i.e., the following identities

$$(\varepsilon_A \otimes \varepsilon_H) \circ \Psi = \varepsilon_H \otimes \varepsilon_A, \quad \delta_{A\otimes H} \circ \Psi = (\Psi \otimes \Psi) \circ \delta_{H\otimes A},$$

hold.

Definition

Let H, A be Hopf quasigroups and let $\Psi : H\otimes A \to A\otimes H$ be a comonoidal distributive law of H over A. We will say that Ψ is an *a*-comonoidal distributive law of H over A if the following identities

$$(A\otimes \mu_H)\circ(\Psi\otimes \mu_H)\circ(H\otimes \Psi\otimes H)\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes H) = \varepsilon_H\otimes A\otimes H,$$

$$(A\otimes \mu_H)\circ(\Psi\otimes \mu_H)\circ(H\otimes \Psi\otimes H)\circ(((H\otimes \lambda_H)\circ\delta_H)\otimes A\otimes H) = \varepsilon_H\otimes A\otimes H,$$

$$(\mu_A\otimes H)\circ(\mu_A\otimes \Psi)\circ(A\otimes \Psi\otimes A)\circ(A\otimes H\otimes((\lambda_A\otimes A)\circ\delta_A)) = A\otimes H\otimes \varepsilon_A,$$

$$(\mu_A\otimes H)\circ(\mu_A\otimes \Psi)\circ(A\otimes \Psi\otimes A)\circ(A\otimes H\otimes((A\otimes \lambda_A)\circ\delta_A)) = A\otimes H\otimes \varepsilon_A,$$

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$$(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((A \otimes \lambda_A) \circ \delta_A)) = A \otimes H \otimes \varepsilon_A,$$

hold.

Note that, if H and A are Hopf algebras and $\Psi : H \otimes A \to A \otimes H$ is a distributive law between the monoids H and A, the previous equalities always hold.

Theorem

Let A and H be Hopf quasigroups. Let $\Psi : H \otimes A \to A \otimes H$ be an a-comonoidal distributive law of H over A. Then the wreath product $A \otimes_{\Psi} H$ built on $A \otimes H$ with tensor unit, counit, coproduct and with the product and antipode defined by

$$\mu_{A\otimes_{\Psi}H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H),$$

and

$$\lambda_{A\otimes_{\Psi}H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H},$$

is a Hopf quasigroup.

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is a Hopf quasigroup.

Therefore, thanks to the previous theorem, we can assert that a-comonoidal distributive laws induce a Hopf quasigroup structure on the tensor product of two Hopf quasigroups. Now, we could also ask under what conditions a Hopf quasigroup structure defined on the tensor product of two Hopf quasigroups is induced by an a-comonoidal distributive law. The following result will address this question.

Theorem

Let A and H be Hopf quasigroups. Suppose that

$$A\odot H=(A\otimes H,\eta_{A\odot H}=\eta_{A\otimes H},\mu_{A\odot H},\varepsilon_{A\odot H}=\varepsilon_{A\otimes H},\delta_{A\odot H}=\delta_{A\otimes H},\lambda_{A\odot H})$$

is a Hopf quasigroup. If the following equalities hold

$$\mu_{A\odot H}=(\mu_A\otimes H)\circ(A\otimes(\mu_{A\odot H}\circ(\eta_A\otimes H\otimes A\otimes H))),$$

$$\mu_{A\odot H}=(A\otimes\mu_H)\circ((\mu_{A\odot H}\circ(A\otimes H\otimes A\otimes\eta_H))\otimes H),$$

$$\mu_{A\odot H}\circ((\mu_{A\odot H}\circ(\eta_A\otimes\lambda_H\otimes\lambda_A\otimes\eta_H))\otimes A\otimes\eta_H)$$
$$=\mu_{A\odot H}\circ(\eta_A\otimes\lambda_H\otimes(\mu_{A\odot H}\circ(\lambda_A\otimes\eta_H\otimes A\otimes\eta_H)),$$

$$\mu_{A\odot H}\circ(\mu_{A\odot H}\circ(\eta_A\otimes H\otimes\eta_A\otimes\lambda_H))\otimes\lambda_A\otimes\eta_H)$$
$$=\mu_{A\odot H}\circ(\eta_A\otimes H\otimes\mu_{A\odot H}\circ(\eta_A\otimes\lambda_H\otimes\lambda_A\otimes\eta_H)),$$

$$\lambda_{A\odot H}\circ(\eta_A\otimes H)=\eta_A\otimes\lambda_H,$$

$$\lambda_{A\odot H}\circ(A\otimes\eta_H)=\lambda_A\otimes\eta_H,$$

then the morphism

$$\Gamma=\mu_{A\odot H}\circ(\eta_A\otimes H\otimes A\otimes\eta_H)$$

is an a-comonoidal distributive law and $A\odot H=A\otimes_\Gamma H$ as Hopf quasigroups.

Examples of distributive laws for Hopf quasigroups

Factorizations and double cross products of Hopf quasigroups

Example

In this example we will see that R-smash product of Hopf quasigroups introduced in

> **T. Brzeziński, Z. Jiao**: R-smash products of Hopf quasigroups, Arabian J. Math., 1 (2012), 39-46.

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is induced by a a-comonoidal distributive law.

Let A, H be Hopf quasigroups in $\mathcal{C}$ with antipodes λ_A, λ_H, respectively. Let

$$R : H \otimes A \to A \otimes H$$

be a morphism satisfying the following conditions:

$$(\varepsilon_A \otimes H) \circ R = H \otimes \varepsilon_A,$$

$$R \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes R) \circ (R \otimes A).$$

Define the R-smash product of A and H, denoted by $A \rtimes_R H$, as

$$A \rtimes_R H = (A \otimes H, \eta_{A \rtimes_R H}, \mu_{A \rtimes_R H}, \varepsilon_{A \rtimes_R H}, \delta_{A \rtimes_R H}, \lambda_{A \rtimes_R H})$$

where

$$\eta_{A \rtimes_R H} = \eta_{A \otimes H}, \quad \varepsilon_{A \rtimes_R H} = \varepsilon_{A \otimes H}, \quad \delta_{A \rtimes_R H} = \delta_{A \otimes H}$$

and

$$\mu_{A \rtimes_R H} = (\mu_A \otimes \mu_H) \circ (A \otimes R \otimes H),$$

$$\lambda_{A \rtimes_R H} = R \circ (\lambda_H \otimes \lambda_H) \circ c_{A,H}.$$

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where

$$\eta_{A \rtimes_R H} = \eta_{A \otimes H}, \quad \varepsilon_{A \rtimes_R H} = \varepsilon_{A \otimes H}, \quad \delta_{A \rtimes_R H} = \delta_{A \otimes H}$$

and

$$\mu_{A \rtimes_R H} = (\mu_A \otimes \mu_H) \circ (A \otimes R \otimes H),$$

$$\lambda_{A \rtimes_R H} = R \circ (\lambda_H \otimes \lambda_H) \circ c_{A,H}.$$

Theorem (T. Brzeziński, Z. Jiao)

$A \rtimes_R H$ is a Hopf quasigroup if, and only if,

- R is a comonoid morphism.
- $R \circ (H \otimes \eta_A) = \eta_A \otimes H, \quad R \circ (\eta_H \otimes A) = A \otimes \eta_H.$
- $R \circ ((\mu_H \circ (H \otimes \lambda_H)) \otimes A) = (A \otimes \mu_H) \circ (R \otimes H) \circ (H \otimes (R \circ (\lambda_H \otimes A)).$
- $(A \otimes \varepsilon_H) \circ R \circ c_{A,H} \circ (A \otimes \lambda_H) \circ R = \varepsilon_H \otimes A.$

Then, if $A \rtimes_R H$ is a Hopf quasigroup, it is easy to show that

$$\Gamma = \mu_{A \rtimes_R H} \circ (\eta_A \otimes H \otimes A \otimes \eta_H) = R$$

is an a-comonoidal distributive law and

$$A \rtimes_R H = A \otimes_R H$$

as Hopf quasigroups.

For the following examples, it is necessary to introduce some additional notions.

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Definition

Let H be a Hopf quasigroup. The pair (M, φ_M) is said to be a left H-module if M is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ (called the action) satisfying

$$\varphi_M \circ (\eta_H \otimes M) = id_M$$

and

$$\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M).$$

Given two left H-modules (M, φ_M), (N, φ_N) and a morphism $f : M \to N$ in $\mathcal{C}$, we will say that f is a morphism of left H-modules if

$$\varphi_N \circ (H \otimes f) = f \circ \varphi_M.$$

Factorizations and double cross products of Hopf quasigroups

We denote the category of left H-modules by ${}_H\mathcal{C}$. It is easy to prove that, if (M, φ_M) and (N, φ_N) are left H-modules, the tensor product $M \otimes N$ is a left H-module with the diagonal action

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

This makes the category of left H-modules into a strict monoidal category $({}_H\mathcal{C}, \otimes, K)$.

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This makes the category of left H-modules into a strict monoidal category $({}_H\mathcal{C}, \otimes, K)$.

Definition

Let H be a Hopf quasigroup. A comonoid A is a left H-module comonoid if it is a left H-module with action φ_A and

$$\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A,$$

$$\delta_A \circ \varphi_A = \varphi_{A\otimes A} \circ (H \otimes \delta_A),$$

hold, i.e., ε_A and δ_A are module morphisms.

Example

In this example we will show that the theory of double cross products of Hopf quasigroups in a symmetric setting, introduced in

> **J.N. Alonso Álvarez, J.M. Fernández Vilaboa y R. González Rodríguez**: Multiplication alteration by two-cocycles. The non-associative version, Bull. Malays. Math. Sci. Soc. 43 (2020), 3557-3615.

produces examples of a-comonoidal distributive laws.

Theorem

Let A, H be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes λ_A, λ_H, respectively. Let (A, φ_A) be a left H-module comonoid, let (H, ϕ_H) be a right A-module comonoid and $\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}$. The following assertions are equivalent:

Factorizations and double cross products of Hopf quasigroups

Theorem

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(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product

$$\mu_{A \bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H)$$

and tensor product unit, counit and coproduct, is a Hopf quasigroup with antipode

$$\lambda_{A \bowtie H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

Theorem

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$$\lambda_{A \bowtie H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

(ii) The equalities

- $\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \quad \phi_H \circ (\eta_H \otimes A) = \eta_H \otimes \varepsilon_A,$
- $(\phi_H \otimes \varphi_A) \circ \delta_{H \otimes A} = c_{A,H} \circ \Psi,$
- $\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ ((\Psi \circ (\lambda_H \otimes \lambda_A)) \otimes A),$
- $\mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes \Psi \otimes H) \circ (\delta_H \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, ,$
- $\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H,$
- $\phi_H \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = \mu_H \circ (\phi_H \otimes H) \circ (H \otimes (\Psi \circ (\lambda_H \otimes \lambda_A))),$
- $\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes \lambda_A) \circ (A \otimes H \otimes \delta_A) = A \otimes \varepsilon_H \otimes \varepsilon_A,$
- $\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A)) = A \otimes \varepsilon_H \otimes \varepsilon_A.$

hold.

Definition

If the conditions of (ii) of the previous theorem hold, we will say that (A, H) is a matched pair of Hopf quasigroups.

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Theorem

If (A, H) is a matched pair of Hopf quasigroups, then

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}$$

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Theorem

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Note that in the Hopf algebra setting, the theorem of the previous slide is the non-associative version of the result proved by Majid in

S. Majid: Foundations of Quantum Group Theory, Cambridge University Press 1995.

for double cross products of Hopf algebras.

Example

In this example, following Sections 4 and 5 of

> **J.N. Alonso Álvarez, J.M. Fernández Vilaboa y R. González Rodríguez**: Multiplication alteration by two-cocycles. The non-associative version, Bull. Malays. Math. Sci. Soc. 43 (2020), 3557-3615,

we provide examples of a-comonoidal distributive laws associated to skew parings between Hopf quasigroups.

Factorizations and double cross products of Hopf quasigroups

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we provide examples of a-comonoidal distributive laws associated to skew parings between Hopf quasigroups.

Definition

Let A, H be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$. A skew pairing between A and H over K is a morphism $\tau : A \otimes H \to K$ satisfying the equalities

- $\tau \circ (\mu_A \otimes H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H)$
- $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H)$,
- $\tau \circ (A \otimes \eta_H) = \varepsilon_A$,
- $\tau \circ (\eta_A \otimes H) = \varepsilon_H$.

If $\tau : A \otimes H \to K$ is a skew pairing, we have that τ is convolution invertible and

$$\tau^* = \tau \circ (\lambda_A \otimes H).$$

Moreover, the following hold:

- $\tau^* \circ (A \otimes \mu_H) = (\tau^* \otimes \tau^*) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H)$
- $\tau^* \circ (\mu_A \otimes H) = (\tau^* \otimes \tau^*) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes (c_{H,H} \circ \delta_H))$
- $\tau^* \circ (A \otimes \eta_H) = \varepsilon_A$,
- $\tau^* \circ (\eta_A \otimes H) = \varepsilon_H$.

In

> **X. Fang, B. Torrecillas**: Twisted smash products and L-R smash products for biquasimodule Hopf quasigroups, Comm. Algebra 42 (2014), 4204-4234,

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Theorem (X. Fang, B. Torrecillas)

Let A, H be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes λ_A, λ_H, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then

$$A \bowtie_\tau H = (A \otimes H, \eta_{A\bowtie_\tau H}, \mu_{A\bowtie_\tau H}, \varepsilon_{A\bowtie_\tau H}, \delta_{A\bowtie_\tau H}, \lambda_{A\bowtie_\tau H})$$

is a Hopf quasigroup where:

$$\eta_{A\bowtie_\tau H} = \eta_A \otimes \eta_H, \quad \varepsilon_{A\bowtie_\tau H} = \varepsilon_A \otimes \varepsilon_H, \quad \delta_{A\bowtie_\tau H} = \delta_{A\otimes H},$$

$$\mu_{A\bowtie_\tau H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H), \quad \lambda_{A\bowtie_\tau H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}$$

$$\Psi = (\tau \otimes A \otimes H \otimes \tau^{-1}) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H} \circ c_{A,A}.$$

Theorem

Let A, H be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes λ_A, λ_H, respectively. Let $\tau : A\otimes H \to K$ be a skew pairing. If we define

$$\varphi_A = (\tau\otimes A\otimes \tau^{-1})\circ (A\otimes H\otimes \delta_A\otimes H)\circ \delta_{A\otimes H}\circ c_{H,A} : H\otimes A\to A$$

and

$$\phi_H = (\tau\otimes H\otimes \tau^{-1})\circ (A\otimes H\otimes c_{A,H}\otimes H)\circ (A\otimes H\otimes A\otimes \delta_H)\circ \delta_{A\otimes H}\circ c_{H,A} : H\otimes A\to H,$$

then (A, H) is a matched pair of Hopf quasigroups and

$$\Psi = (\tau\otimes A\otimes H\otimes \tau^{-1})\circ (A\otimes H\otimes \delta_{A\otimes H})\circ \delta_{A\otimes H}\circ c_{A,A} = (\varphi_A\otimes \phi_H)\circ \delta_{H\otimes A}$$

is an a-comonoidal distributive law.
Therefore, $A\bowtie_\tau H$ is the double cross product of Hopf quasigroups associated to (A, H).

Example

This example is a particular case of the previous one. Let $\mathbb{F}$ be a field such that Char($\mathbb{F}$)$\neq$ 2 and denote the tensor product over $\mathbb{F}$ as $\otimes$. Consider the non-abelian group $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$, where σ_0 is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let u be an additional element such that $u^2 = 1$.
By the results proved in

> **O. Chein**: Moufang loops of small order I, Trans. Amer. Math. Soc. 188 (1974) 31-51,

we know that the set

$$L = M(S_3, 2) = \{\sigma_i u^{\alpha} \ ; \ \alpha = 0, 1\}$$

is a Moufang loop where the product is defined by

$$\sigma_i u^{\alpha} \bullet \ \sigma_j u^{\beta} = (\sigma_i^{\nu} \sigma_j^{\mu})^{\nu} u^{\alpha+\beta}, \ \ \nu = (-1)^{\beta}, \ \mu = (-1)^{\alpha+\beta}.$$

Then, L is an IP loop and $\mathbb{F}[L]$ is a cocommutative Hopf quasigroup.

Let H_4 be the 4-dimensional Sweedler Hopf algebra. The basis of H_4 is $\{1, x, y, w = xy\}$ and the multiplication table is defined by

	x	y	w
x	1	w	y
y	$-w$	0	0
w	$-y$	0	0

The costructure of H_4 is given by

$$\delta_{H_4}(x) = x \otimes x,\ \delta_{H_4}(y) = y \otimes x + 1 \otimes y,\ \delta_{H_4}(w) = w \otimes 1 + x \otimes w,$$

$$\varepsilon_{H_4}(x) = 1_{\mathbb{F}},\ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0$$

and the antipode λ_{H_4} is described by

$$\lambda_{H_4}(x) = x,\ \lambda_{H_4}(y) = w,\ \lambda_{H_4}(w) = -y.$$

The morphism $\tau : \mathbb{F}[L] \otimes H_4 \to \mathbb{F}$ defined by

$$\tau(\sigma_i u^\alpha \otimes z) = \begin{cases} 1 & \text{si} \quad z = 1 \\ (-1)^\alpha & \text{si} \quad z = x \\ 0 & \text{si} \quad z = y, w \end{cases}$$

is a skew pairing. Then, $\mathbb{F}[L] \bowtie_\tau H_4$ is a Hopf quasigroup asociated to the a-comonoidal distributive law Ψ of $\mathbb{F}[L]$ over H_4 where

$$\Psi(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes 1, \;\; \Psi(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes x,$$

$$\Psi(y \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes y, \;\; \Psi(w \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes w.$$

Factorizations and double cross products of Hopf quasigroups

The morphism $\tau : \mathbb{F}[L] \otimes H_4 \to \mathbb{F}$ defined by

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More concretely, $\mathbb{F}[L] \bowtie_\tau H_4 = \mathbb{F}[L] \bowtie H_4$ for the matched pair $(\mathbb{F}[L], H_4)$ where the actions are:

$$\varphi_{\mathbb{F}[L]}(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha, \; \varphi_{\mathbb{F}[L]}(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha, \; \varphi_{\mathbb{F}[L]}(y \otimes \sigma_i u^\alpha) = \varphi_{\mathbb{F}[L]}(w \otimes \sigma_i u^\alpha) = 0$$

and

$$\phi_{H_4}(1 \otimes \sigma_i u^\alpha) = 1, \;\; \phi_{H_4}(x \otimes \sigma_i u^\alpha) = x, \;\; \phi_{H_4}(y \otimes \sigma_i u^\alpha) = (-1)^\alpha y,$$

$$\phi_{H_4}(w \otimes \sigma_i u^\alpha) = (-1)^\alpha w.$$

Theorem

$\mathbb{F}[M(S_3, 2)] \bowtie_\tau H_4$ is a Hopf quasigroup that is neither commutative nor cocommutative.

Factorizations of Hopf quasigroups

A Hopf algebra X in $\mathbb{F}$-Vect factorises as

$$X = AH$$

if A and H are sub-Hopf algebras of X, with inclusion maps i_A and i_H, such that the map

$$\omega(a \otimes h) = i_A(a) i_H(h)$$

is an isomorphism of vector spaces.

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As was proved by Majid in

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The main objective of this final section is to extend this result to the theory of factorizations of Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$.

Definition

Let X be a Hopf quasigroup in C. Let H, A be Hopf subquasigroups of X with inclusion morphisms $i_H : H \to X$, $i_A : A \to X$ respectively. Let ω_X and θ_X be the morphisms defined by

$$\omega_X = \mu_X \circ (i_A \otimes i_H) : A \otimes H \to X, \quad \theta_X = \mu_X \circ (i_H \otimes i_A) : H \otimes A \to X.$$

We will say that X factorizes as

$$X = AH$$

if ω_X is an isomorphism and the following identities

$$\mu_X \circ (\omega_X \otimes X) = \mu_X \circ (i_A \otimes (\mu_X \circ (i_H \otimes X))),$$

$$\mu_X \circ (X \otimes \omega_X) = \mu_X \circ ((\mu_X \circ (X \otimes i_A)) \otimes i_H),$$

$$\mu_X \circ ((\theta_X \circ (\lambda_H \otimes \lambda_A)) \otimes X) = \mu_X \circ ((i_H \circ \lambda_H) \otimes (\mu_X \circ ((i_A \circ \lambda_A) \otimes X))),$$

$$\mu_X \circ (X \otimes (\theta_X \circ (\lambda_H \otimes \lambda_A))) = \mu_X \circ ((\mu_X \circ (X \otimes (i_H \circ \lambda_H)) \otimes (i_A \circ \lambda_A)),$$

hold.

Then, if the antipodes of H and A are isomorphisms, then we can remove the antipodes in the factorization definition. Then the two last identities become in

$$\mu_X \circ (\theta_X \otimes X) = \mu_X \circ (i_H \otimes (\mu_X \circ (i_A \otimes X)),$$

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Example

Suppose that (A, H) is a matched pair of Hopf quasigroups. The double cross product

$$A \bowtie H$$

is a Hopf quasigroup.
The morphisms $i_A = A \otimes \eta_H : A \to A \bowtie H$ and $i_H = \eta_A \otimes H : H \to A \bowtie H$ are morphisms of Hopf quasigroups.
Also, we obtain that

$$\omega_{A \bowtie H} = id_{A \otimes H}, \quad \theta_{A \bowtie H} = \Psi$$

and therefore any matched pair of Hopf quasigroups induces an example of factorization.

Theorem

Let H, A be Hopf subquasigroups of a Hopf quasigroup X. If X factorises as $X = AH$, the morphism

$$\Psi = \omega_X^{-1} \circ \theta_X : H \otimes A \to A \otimes H$$

is a comonoidal distributive law of H over A. Moreover, if the antipodes of H and A are isomorphisms Ψ is an a-comonoidal distributive law of H over A.

Theorem

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Theorem

Let H, A be Hopf subquasigroups of a Hopf quasigroup X such that the antipodes of H and A are isomorphisms. Assume that X factorises as $X = AH$. Then, ω_X is an isomorphism of Hopf quasigrous between the wreath product $A \otimes_\Psi H$ and X, where Ψ is the *a*-comonoidal distributive law defined in the previous theorem.

Theorem

Let H, A, X be Hopf quasigroups such that the antipodes of H and A are isomorphisms. If X factorizes as $X = AH$, there exists a matched pair of Hopf quasigroups (A, H) such that X is isomorphic to $A \bowtie H$ as Hopf quasigroups.

Factorizations and double cross products of Hopf quasigroups

Theorem

Let H, A, X be Hopf quasigroups such that the antipodes of H and A are isomorphisms. If X factorizes as $X = AH$, there exists a matched pair of Hopf quasigroups (A, H) such that X is isomorphic to $A \bowtie H$ as Hopf quasigroups.

Proof

Let Ψ be the morphism defined in the previous theorems. Define the actions by

$$\varphi_A = (A \otimes \varepsilon_H) \circ \Psi, \quad \phi_H = (\varepsilon_A \otimes H) \circ \Psi.$$

Theorem

Let H, A, X be Hopf quasigroups such that the antipodes of H and A are isomorphisms. Then, X factorizes as $X = AH$ if, and only if, there exists a matched pair of Hopf quasigroups (A, H) such that X is isomorphic to $A \bowtie H$ as Hopf quasigroups.

Factorizations and double cross products of Hopf quasigroups

Theorem

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Thank you

Factorizations and double cross products of Hopf quasigroups