# Wreath products and factorizations for Hopf quasigroups

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# Hopf quasigroups

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Recall that a monoidal category is a category C together with a functor

$$\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$$

called tensor product, an object $K$ of C, called the unit object, and families of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), \quad r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in C, called associativity, right unit and left unit constraints, respectively, satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where for each object $X$ in C, $id_X$ denotes the identity morphism of $X$.

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A monoidal category is called strict if the constraints of the previous paragraph are identities.

It is a well-known fact that every non-strict monoidal category is monoidal equivalent to a strict one. This lets us to treat monoidal categories as if they were strict and, as a consequence, the results proved in a strict setting hold for every non-strict monoidal category.

A braiding for a strict monoidal category C is a natural family of isomorphisms

$$c_{M,N} : M \otimes N \to N \otimes M$$

subject to the conditions

$$c_{M,N\otimes P} = (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P), \quad c_{M\otimes N,P} = (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P})$$

for all $M, N, P \in$ C.


Wreath products and factorizations for Hopf quasigroups

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If

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for all $M, N$ in C, we will say that C is symmetric.

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**Notation**

For simplicity of notation, given objects $M$, $N$, $P$ in C and a morphism $f : M \to N$, we will write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

### Definition

A magma in $\mathcal{C}$ is a pair $A = (A, \mu_A)$, where $A$ is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ (product) is a morphism in $\mathcal{C}$.
A unital magma in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$, where $(A, \mu_A)$ is a magma in $\mathcal{C}$ and $\eta_A : K \to A$ (unit) is a morphism in $\mathcal{C}$ such that

$$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A).$$

A monoid in $\mathcal{C}$ is a unital magma $A = (A, \eta_A, \mu_A)$ in $\mathcal{C}$ satisfying

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Given two unital magmas (monoids) $A$ and $B$, $f : A \to B$ is a morphism of unital magmas (monoids) if $f \circ \eta_A = \eta_B$ and $\mu_B \circ (f \otimes f) = f \circ \mu_A$.

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If $A$, $B$ are are unital magmas (monoids), $A \otimes B$ is also a unital magma (monoid) where

$$\eta_{A\otimes B} = \eta_A \otimes \eta_B, \quad \mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B).$$

### Definition

A comagma in $\mathcal{C}$ is a pair $D = (D, \delta_D)$, where $D$ is an object in $\mathcal{C}$ and $\delta_D : D \to D \otimes D$ (coproduct) is a morphism in $\mathcal{C}$. A counital comagma in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where $(D, \delta_D)$ is a comagma in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit) is a morphism in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. A comonoid in $\mathcal{C}$ is a counital comagma in $\mathcal{C}$ satisfying $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, i.e., the coproduct $\delta_D$ is coassociative.

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## Definition

If $D$ and $E$ are counital comagmas (comonoids) in $\mathcal{C}$, $f : D \to E$ is a morphism of counital comagmas (comonoids) if $\varepsilon_E \circ f = \varepsilon_D$, and $(f \otimes f) \circ \delta_D = \delta_E \circ f$.


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Moreover, if $D$, $E$ are counital comagmas (comonoids), $D \otimes E$ is a counital comagma (comonoid), where

$$\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E, \quad \delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E).$$


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## Definition

Let $f : D \to A$ and $g : D \to A$ be morphisms between a comagma $D$ and a magma $A$. We define the convolution product by

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_D.$$

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$$f * g = \mu_A \circ (f \otimes g) \circ \delta_D.$$

If $A$ is unital and $D$ counital, we will say that $f$ is convolution invertible if there exists $f^* : D \to A$ such that

$$f * f^* = f^* * f = \varepsilon_D \otimes \eta_A.$$

**Definition**

A non-associative bimonoid in the category $\mathcal{C}$ is a unital magma $(H, \eta_H, \mu_H)$ and a comonoid $(H, \varepsilon_H, \delta_H)$ such that $\varepsilon_H$ and $\delta_H$ are morphisms of unital magmas (equivalently, $\eta_H$ and $\mu_H$ are morphisms of counital comagmas). Then the following identities hold:

$$\varepsilon_H \circ \eta_H = id_K, \quad \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H,$$

$$\delta_H \circ \eta_H = \eta_H \otimes \eta_H, \quad \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H}.$$

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### Definition

A non-associative bimonoid is called cocommutative if $\delta_H = c_{H,H} \circ \delta_H$.

## Definition

**J. Klim, S. Majid**: Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323 (2010), 3067-3110. ($\mathcal{C} = \mathbb{F}$-Vect)

A Hopf quasigroup $H$ in $\mathcal{C}$ is a non-associative bimonoid such that there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of $H$) satisfying

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H)$$

and

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H).$$

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and

$$\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H).$$

Note that composing with $H \otimes \eta_H$ in the first equality we obtain that

$$\lambda_H * id_H = \varepsilon_H \otimes \eta_H,$$

and composing with $\eta_H \otimes H$ in the second one we obtain

$$id_H * \lambda_H = \varepsilon_H \otimes \eta_H.$$

Therefore, $\lambda_H$ is convolution invertible and $\lambda_H^* = id_H$.

If $H$ is a Hopf quasigroup in $\mathcal{C}$ we know that the antipode $\lambda_H$ is unique.


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If $H$ is a Hopf quasigroup in $\mathcal{C}$ we know that the antipode $\lambda_H$ is unique.

Also, the antipode is antimultiplicative and anticomultiplicative, i.e.,

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H),$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H$$

and leaves the unit and the counit invariable:

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

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### Definition

A morphism between Hopf quasigroups $H$ and $B$ is a morphism $f : H \to B$ of unital magmas and comonoids, i.e., a morphism of non-associative bimonoids.
Then the equality

$$\lambda_B \circ f = f \circ \lambda_H$$

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holds.

**A Hopf quasigroup $H$ is associative if, and only if, $H$ is a Hopf monoid (algebra).**

## Example

Suppose that $\mathcal{C} = \mathrm{Set}$. Then $L$ is a Hopf quasigroup in $\mathcal{C}$ if, and only if, $L$ is an IP loop. An IP loop is a set $L$ with a product, identity $e_L$, and with the property that for each $u \in L$ there exists $u^{-1} \in L$ (the inverse of $u$) such that

$$u^{-1}(uv) = v, \quad (vu)u^{-1} = v, \quad \forall \ v \in L.$$

As a consequence, it is easy to show that , if $L$ is an IP loop, for all $u \in L$ the element $u^{-1}u = e_L = uu^{-1}$, $u^{-1}$ is unique and $(u^{-1})^{-1} = u$. Moreover, $(uv)^{-1} = v^{-1}u^{-1}$ holds for any pair of elements $u, v \in L$.
Note that in this case $L$ is a cocommutative Hopf quasigroup because

$$\delta_L(u) = (u, u).$$


Wreath products and factorizations for Hopf quasigroups

## Example

Suppose that $\mathcal{C}$ is $\mathbb{F}$-Vect. Let $L$ be an IP loop. Then, the loop algebra

$$\mathbb{F}[L] = \bigoplus_{u \in L} \mathbb{F}u$$

is a cocommutative non-associative bimonoid with unit $\eta_{\mathbb{F}[L]}(1_{\mathbb{F}}) = e_L$, product defined by linear extension of the one defined in $L$, and coproduct and counit

$$\delta_{\mathbb{F}[L]}(u) = u \otimes u, \quad \varepsilon_{\mathbb{F}[L]}(u) = 1_{\mathbb{F}}.$$

Also, it is a Hopf quasigroup where the antipode is defined by the linear extension of the map

$$\lambda_{\mathbb{F}[L]}(u) = u^{-1}.$$

## Example

Let $\mathbb{F}$ be a field. A Malcev algebra over $\mathbb{F}$ is an anticommutative algebra $(M, [,])$ such that

$$[J(a,b,c),a] = J(a,b,[a,c]),$$

where $J(a,b,c) = [[a,b],c] - [[a,c],b] - [a,[b,c]]$ denotes the Jacobian in $a$, $b$, $c$. As was proved in

> **J. Klim, S. Majid**: Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323 (2010), 3067-3110,

if the characteristic of $\mathbb{F}$ is different of 2 and 3, then the universal enveloping algebra $U(M)$, introduced by

> **J.M. Pérez Izquierdo, I.P. Shestakov**: An envelope for Malcev algebras, J. Algebra 272 (2004), 379-393,

admits a cocommutative Hopf quasigroup structure.

# Distributive laws for Hopf quasigroups

## Definition

Let $H$, $A$ be Hopf quasigroups. A morphism

$$\Psi : H \otimes A \to A \otimes H$$

is said to be a distributive law of $H$ over $A$ if the following identities

$$\Psi \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A) \circ (\lambda_H \otimes \lambda_A \otimes A),$$

$$\Psi \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = (A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi) \circ (H \otimes \lambda_H \otimes \lambda_A),$$

$$\Psi \circ (H \otimes \eta_A) = \eta_A \otimes H, \quad \Psi \circ (\eta_H \otimes A) = A \otimes \eta_H,$$

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$$\Psi \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = (A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi) \circ (H \otimes \lambda_H \otimes \lambda_A),$$

$$\Psi \circ (H \otimes \eta_A) = \eta_A \otimes H, \quad \Psi \circ (\eta_H \otimes A) = A \otimes \eta_H,$$

hold.

If the antipodes of $H$ and $A$ are isomorphisms, the two first identities are equivalent to

$$\Psi \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A),$$

$$\Psi \circ (\mu_H \otimes A) = (A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi),$$

respectively. Then, in this case, the conditions of the definition of distributive law for Hopf quasigroups are the ones that we can find in the classical definition of distributive law between monoids, i.e., $\Psi$ is compatible with the unit and the product of $A$ and $H$.

**Definition**

Let $H$, $A$ be Hopf quasigroups and let $\Psi : H \otimes A \to A \otimes H$ be a distributive law of $H$ over $A$. The distributive law $\Psi$ is said to be comonoidal if it is a comonoid morphism, i.e., the following identities

$$(\varepsilon_A \otimes \varepsilon_H) \circ \Psi = \varepsilon_H \otimes \varepsilon_A, \quad \delta_{A\otimes H} \circ \Psi = (\Psi \otimes \Psi) \circ \delta_{H\otimes A},$$

hold.


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### Definition

Let $H$, $A$ be Hopf quasigroups and let $\Psi : H \otimes A \to A \otimes H$ be a comonoidal distributive law of $H$ over $A$. We will say that $\Psi$ is an *a*-comonoidal distributive law of $H$ over $A$ if the following identities

$$(A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ (((\lambda_H \otimes H) \circ \delta_H) \otimes A \otimes H) = \varepsilon_H \otimes A \otimes H,$$

$$(A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H) = \varepsilon_H \otimes A \otimes H,$$

$$(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A)) = A \otimes H \otimes \varepsilon_A,$$

$$(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((A \otimes \lambda_A) \circ \delta_A)) = A \otimes H \otimes \varepsilon_A,$$

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Wreath products and factorizations for Hopf quasigroups

### Definition

Let $H$, $A$ be Hopf quasigroups and let $\Psi : H\otimes A \to A\otimes H$ be a comonoidal distributive law of $H$ over $A$. We will say that $\Psi$ is an *a*-comonoidal distributive law of $H$ over $A$ if the following identities

$$(A\otimes \mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes H) = \varepsilon_H\otimes A\otimes H,$$

$$(A\otimes \mu_H)\circ(\Psi\otimes\mu_H)\circ(H\otimes\Psi\otimes H)\circ(((H\otimes \lambda_H)\circ\delta_H)\otimes A\otimes H) = \varepsilon_H\otimes A\otimes H,$$

$$(\mu_A\otimes H)\circ(\mu_A\otimes\Psi)\circ(A\otimes\Psi\otimes A)\circ(A\otimes H\otimes((\lambda_A\otimes A)\circ\delta_A)) = A\otimes H\otimes\varepsilon_A,$$

$$(\mu_A\otimes H)\circ(\mu_A\otimes\Psi)\circ(A\otimes\Psi\otimes A)\circ(A\otimes H\otimes((A\otimes\lambda_A)\circ\delta_A)) = A\otimes H\otimes\varepsilon_A,$$

hold.

Note that, if $H$ and $A$ are Hopf algebras and $\Psi : H\otimes A\to A\otimes H$ is a distributive law between the monoids $H$ and $A$, the previous equalities always hold.

### Theorem

Let $A$ and $H$ be Hopf quasigroups. Let $\Psi : H \otimes A \to A \otimes H$ be an *a*-comonoidal distributive law of $H$ over $A$. Then the wreath product $A \otimes_{\Psi} H$ built on $A \otimes H$ with tensor unit, counit, coproduct and with the product and antipode defined by

$$\mu_{A \otimes_{\Psi} H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H),$$

and

$$\lambda_{A \otimes_{\Psi} H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H},$$

is a Hopf quasigroup.

## Theorem

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is a Hopf quasigroup.

Therefore, thanks to the previous theorem, we can assert that a-comonoidal distributive laws induce a Hopf quasigroup structure on the tensor product of two Hopf quasigroups. Now, we could also ask under what conditions a Hopf quasigroup structure defined on the tensor product of two Hopf quasigroups is induced by an a-comonoidal distributive law. The following result will address this question.

## Theorem

Let $A$ and $H$ be Hopf quasigroups. Suppose that

$$A \odot H = (A \otimes H, \eta_{A\odot H} = \eta_{A\otimes H}, \mu_{A\odot H}, \varepsilon_{A\odot H} = \varepsilon_{A\otimes H}, \delta_{A\odot H} = \delta_{A\otimes H}, \lambda_{A\odot H})$$

is a Hopf quasigroup. If the following equalities hold

$$\mu_{A\odot H} = (\mu_A \otimes H) \circ (A \otimes (\mu_{A\odot H} \circ (\eta_A \otimes H \otimes A \otimes H))),$$

$$\mu_{A\odot H} = (A \otimes \mu_H) \circ ((\mu_{A\odot H} \circ (A \otimes H \otimes A \otimes \eta_H)) \otimes H),$$

$$\mu_{A\odot H} \circ ((\mu_{A\odot H} \circ (\eta_A \otimes \lambda_H \otimes \lambda_A \otimes \eta_H)) \otimes A \otimes \eta_H)$$

$$= \mu_{A\odot H} \circ (\eta_A \otimes \lambda_H \otimes (\mu_{A\odot H} \circ (\lambda_A \otimes \eta_H \otimes A \otimes \eta_H)),$$

$$\mu_{A\odot H} \circ (\mu_{A\odot H} \circ (\eta_A \otimes H \otimes \eta_A \otimes \lambda_H)) \otimes \lambda_A \otimes \eta_H)$$

$$= \mu_{A\odot H} \circ (\eta_A \otimes H \otimes \mu_{A\odot H} \circ (\eta_A \otimes \lambda_H \otimes \lambda_A \otimes \eta_H)),$$

$$\lambda_{A\odot H} \circ (\eta_A \otimes H) = \eta_A \otimes \lambda_H,$$

$$\lambda_{A\odot H} \circ (A \otimes \eta_H) = \lambda_A \otimes \eta_H,$$

then the morphism

$$\Gamma = \mu_{A\odot H} \circ (\eta_A \otimes H \otimes A \otimes \eta_H)$$

is an a-comonoidal distributive law and $A \odot H = A \otimes_\Gamma H$ as Hopf quasigroups.

# Examples of distributive laws for Hopf quasigroups

**Example**

In this example we will see that $R$-smash product of Hopf quasigroups introduced in

> **T. Brzeziński, Z. Jiao**: $R$-smash products of Hopf quasigroups, Arabian J. Math., 1 (2012), 39-46.

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is induced by a a-comonoidal distributive law.

Let $A$, $H$ be Hopf quasigroups in $\mathcal{C}$ with antipodes $\lambda_A$, $\lambda_H$, respectively. Let

$$R : H \otimes A \to A \otimes H$$

be a morphism satisfying the following conditions:

$$(\varepsilon_A \otimes H) \circ R = H \otimes \varepsilon_A,$$

$$R \circ (H \otimes \mu_A) = (\mu_A \otimes H) \circ (A \otimes R) \circ (R \otimes A).$$

Define the $R$-smash product of $A$ and $H$, denoted by $A \rtimes_R H$, as

$$A \rtimes_R H = (A \otimes H, \eta_{A \rtimes_R H}, \mu_{A \rtimes_R H}, \varepsilon_{A \rtimes_R H}, \delta_{A \rtimes_R H}, \lambda_{A \rtimes_R H})$$

where

$$\eta_{A \rtimes_R H} = \eta_{A \otimes H}, \quad \varepsilon_{A \rtimes_R H} = \varepsilon_{A \otimes H}, \quad \delta_{A \rtimes_R H} = \delta_{A \otimes H}$$

and

$$\mu_{A \rtimes_R H} = (\mu_A \otimes \mu_H) \circ (A \otimes R \otimes H),$$

$$\lambda_{A \rtimes_R H} = R \circ (\lambda_H \otimes \lambda_H) \circ c_{A,H}.$$

Define the $R$-smash product of $A$ and $H$, denoted by $A \rtimes_R H$, as

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where

$$\eta_{A \rtimes_R H} = \eta_{A \otimes H}, \quad \varepsilon_{A \rtimes_R H} = \varepsilon_{A \otimes H}, \quad \delta_{A \rtimes_R H} = \delta_{A \otimes H}$$

and

$$\mu_{A \rtimes_R H} = (\mu_A \otimes \mu_H) \circ (A \otimes R \otimes H),$$
$$\lambda_{A \rtimes_R H} = R \circ (\lambda_H \otimes \lambda_H) \circ c_{A,H}.$$

Then Brzeziński and Jiao proved that $A \rtimes_R H$ is a Hopf quasigroup if, and only if,

- $R$ is a comonoid morphism.
- $R \circ (H \otimes \eta_A) = \eta_A \otimes H, \quad R \circ (\eta_H \otimes A) = A \otimes \eta_H.$
- $R \circ ((\mu_H \circ (H \otimes \lambda_H)) \otimes A) = (A \otimes \mu_H) \circ (R \otimes H) \circ (H \otimes (R \circ (\lambda_H \otimes A))).$
- $(A \otimes \varepsilon_H) \circ R \circ c_{A,H} \circ (A \otimes \lambda_H) \circ R = \varepsilon_H \otimes A.$


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Then, if $A \rtimes_R H$ is a Hopf quasigroup, it is easy to show that

$$\Gamma = \mu_{A \rtimes_R H} \circ (\eta_A \otimes H \otimes A \otimes \eta_H) = R$$

is an a-comonoidal distributive law and

$$A \rtimes_R H = A \otimes_R H$$

as Hopf quasigroups.

For the following examples, it is necessary to introduce some additional notions.


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### Definition

Let $H$ be a Hopf quasigroup. The pair $(M, \varphi_M)$ is said to be a left $H$-module if $M$ is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \to M$ is a morphism in $\mathcal{C}$ (called the action) satisfying

$$\varphi_M \circ (\eta_H \otimes M) = id_M$$

and

$$\varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M).$$

Given two left $H$-modules $(M, \varphi_M)$, $(N, \varphi_N)$ and a morphism $f : M \to N$ in $\mathcal{C}$, we will say that $f$ is a morphism of left $H$-modules if

$$\varphi_N \circ (H \otimes f) = f \circ \varphi_M.$$

We denote the category of left $H$-modules by ${}_H\mathcal{C}$. It is easy to prove that, if $(M, \varphi_M)$ and $(N, \varphi_N)$ are left $H$-modules, the tensor product $M \otimes N$ is a left $H$-module with the diagonal action

$$\varphi_{M\otimes N} = (\varphi_M \otimes \varphi_N) \circ (H \otimes c_{H,M} \otimes N) \circ (\delta_H \otimes M \otimes N).$$

This makes the category of left $H$-modules into a strict monoidal category $({}_H\mathcal{C}, \otimes, K)$.

## Definition

Let $H$ be a Hopf quasigroup. We will say that a unital magma $A$ is a left $H$-module magma if it is a left $H$-module with action

$$\varphi_A : H \otimes A \to A$$

and the following equalities

$$\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A,$$

$$\mu_A \circ \varphi_{A \otimes A} = \varphi_A \circ (H \otimes \mu_A),$$

hold, i.e., $\eta_A$ and $\mu_A$ are module morphisms.


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## Definition

Let $H$ be a Hopf quasigroup. A comonoid $A$ is a left $H$-module comonoid if it is a left $H$-module with action $\varphi_A$ and

$$\varepsilon_A \circ \varphi_A = \varepsilon_H \otimes \varepsilon_A,$$

$$\delta_A \circ \varphi_A = \varphi_{A \otimes A} \circ (H \otimes \delta_A),$$

hold, i.e., $\varepsilon_A$ and $\delta_A$ are module morphisms.


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**Example**

In this example we will show that the theory of double cross products of Hopf quasigroups in a symmetric setting, introduced in

> **J.N. Alonso Álvarez, J.M. Fernández Vilaboa y R. González Rodríguez**: Multiplication alteration by two-cocycles. The non-associative version, Bull. Malays. Math. Sci. Soc. 43 (2020), 3557-3615.

produces examples of $a$-comonoidal distributive laws.

### Theorem

Let $A$, $H$ be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes $\lambda_A$, $\lambda_H$, respectively. Let $(A, \varphi_A)$ be a left $H$-module comonoid, let $(H, \phi_H)$ be a right $A$-module comonoid and $\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}$. The following assertions are equivalent:


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(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product

$$\mu_{A \bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H)$$

and tensor product unit, counit and coproduct, is a Hopf quasigroup with antipode

$$\lambda_{A \bowtie H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

## Theorem

Let $A$, $H$ be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes $\lambda_A$, $\lambda_H$, respectively. Let $(A, \varphi_A)$ be a left $H$-module comonoid, let $(H, \phi_H)$ be a right $A$-module comonoid and $\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}$. The following assertions are equivalent:

(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product

$$\mu_{A \bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H)$$

and tensor product unit, counit and coproduct, is a Hopf quasigroup with antipode

$$\lambda_{A \bowtie H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}.$$

(ii) The equalities

- $\varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \quad \phi_H \circ (\eta_H \otimes A) = \eta_H \otimes \varepsilon_A,$
- $(\phi_H \otimes \varphi_A) \circ \delta_{H \otimes A} = c_{A,H} \circ \Psi,$
- $\varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ ((\Psi \circ (\lambda_H \otimes \lambda_A)) \otimes A),$
- $\mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes \Psi \otimes H) \circ (\delta_H \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, ,$
- $\mu_H \circ (\phi_H \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H,$
- $\phi_H \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = \mu_H \circ (\phi_H \otimes H) \circ (H \otimes (\Psi \circ (\lambda_H \otimes \lambda_A))),$
- $\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes \lambda_A) \circ (A \otimes H \otimes \delta_A) = A \otimes \varepsilon_H \otimes \varepsilon_A,$
- $\mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \circ \delta_A)) = A \otimes \varepsilon_H \otimes \varepsilon_A.$

hold.

**Definition**

If the conditions of (ii) of the previous theorem hold, we will say that $(A, H)$ is a matched pair of Hopf quasigroups.


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**Theorem**

If $(A, H)$ is a matched pair of Hopf quasigroups, then

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H\otimes A}$$

is an *a*-comonoidal distributive law.

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is an *a*-comonoidal distributive law.

Note that in the Hopf algebra setting, the theorem of the previous slide is the non-associative version of the result proved by Majid in

> **S. Majid**: Foundations of Quantum Group Theory, Cambridge University Press 1995.

for double cross products of Hopf algebras.

**Example**

In this example, following Sections 4 and 5 of

> **J.N. Alonso Álvarez, J.M. Fernández Vilaboa y R. González Rodríguez**: Multiplication alteration by two-cocycles. The non-associative version, Bull. Malays. Math. Sci. Soc. 43 (2020), 3557-3615,

we provide examples of *a*-comonoidal distributive laws associated to skew parings between Hopf quasigroups.


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we provide examples of $a$-comonoidal distributive laws associated to skew parings between Hopf quasigroups.

## Definition

Let $A$, $H$ be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$. A skew pairing between $A$ and $H$ over $K$ is a morphism $\tau : A \otimes H \to K$ satisfying the equalities

- $\tau \circ (\mu_A \otimes H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H)$
- $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H)$,
- $\tau \circ (A \otimes \eta_H) = \varepsilon_A$,
- $\tau \circ (\eta_A \otimes H) = \varepsilon_H$.

If $\tau : A \otimes H \to K$ is a skew pairing, we have that $\tau$ is convolution invertible and

$$\tau^* = \tau \circ (\lambda_A \otimes H).$$

Moreover, the following hold:

- $\tau^* \circ (A \otimes \mu_H) = (\tau^* \otimes \tau^*) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H)$
- $\tau^* \circ (\mu_A \otimes H) = (\tau^* \otimes \tau^*) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes (c_{H,H} \circ \delta_H))$
- $\tau^* \circ (A \otimes \eta_H) = \varepsilon_A$,
- $\tau^* \circ (\eta_A \otimes H) = \varepsilon_H$.


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In

> **X. Fang, B. Torrecillas**: Twisted smash products and L-R smash products for biquasimodule Hopf quasigroups, Comm. Algebra 42 (2014), 4204-4234.

we can find a result proved in the category of vector spaces that in our monoidal setting is the following:

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we can find a result proved in the category of vector spaces that in our monoidal setting is the following:

**Theorem**

Let $A$, $H$ be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes $\lambda_A$, $\lambda_H$, respectively. Let $\tau : A\otimes H\rightarrow K$ be a skew pairing. Then

$$A\bowtie_{\tau} H=(A\otimes H, \eta_{A\bowtie_{\tau} H}, \mu_{A\bowtie_{\tau} H}, \varepsilon_{A\bowtie_{\tau} H}, \delta_{A\bowtie_{\tau} H}, \lambda_{A\bowtie_{\tau} H})$$

is a Hopf quasigroup where:

$$\eta_{A\bowtie_{\tau} H}=\eta_A\otimes\eta_H,\quad \varepsilon_{A\bowtie_{\tau} H}=\varepsilon_A\otimes\varepsilon_H,\quad \delta_{A\bowtie_{\tau} H}=\delta_{A\otimes H},$$

$$\mu_{A\bowtie_{\tau} H}=(\mu_A\otimes\mu_H)\circ(A\otimes\Psi\otimes H),\quad \lambda_{A\bowtie_{\tau} H}=\Psi\circ(\lambda_H\otimes\lambda_A)\circ c_{A,H}$$

$$\Psi=(\tau\otimes A\otimes H\otimes\tau^{-1})\circ(A\otimes H\otimes\delta_{A\otimes H})\circ\delta_{A\otimes H}\circ c_{A,A}.$$

## Theorem

Let $A$, $H$ be Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$ with antipodes $\lambda_A$, $\lambda_H$, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. If we define

$$\varphi_A = (\tau \otimes A \otimes \tau^{-1}) \circ (A \otimes H \otimes \delta_A \otimes H) \circ \delta_{A \otimes H} \circ c_{H,A} : H \otimes A \to A$$

and

$$\phi_H = (\tau \otimes H \otimes \tau^{-1}) \circ (A \otimes H \otimes c_{A,H} \otimes H) \circ (A \otimes H \otimes A \otimes \delta_H) \circ \delta_{A \otimes H} \circ c_{H,A} :$$

$$H \otimes A \to H,$$

then $(A, H)$ is a matched pair of Hopf quasigroups and

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}$$

is an *a*-comonoidal distributive law.
Therefore, $A \bowtie_\tau H$ is the double cross product of Hopf quasigroups associated to $(A, H)$.

## Example

This example is a particular case of the previous one. Let $\mathbb{F}$ be a field such that Char($\mathbb{F}$)$\neq$ 2 and denote the tensor product over $\mathbb{F}$ as $\otimes$. Consider the non-abelian group $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$, where $\sigma_0$ is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let $u$ be an additional element such that $u^2 = 1$.
By the results proved in

> **O. Chein**: Moufang loops of small order I, Trans. Amer. Math. Soc. 188 (1974) 31-51,

we know that the set

$$L = M(S_3, 2) = \{\sigma_i u^{\alpha} \; ; \; \alpha = 0, 1\}$$

is a Moufang loop where the product is defined by

$$\sigma_i u^{\alpha} \bullet \; \sigma_j u^{\beta} = (\sigma_i^{\nu} \sigma_j^{\mu})^{\nu} u^{\alpha+\beta}, \;\; \nu = (-1)^{\beta}, \; \mu = (-1)^{\alpha+\beta}.$$

Then, $L$ is an IP loop and $\mathbb{F}[L]$ is a cocommutative Hopf quasigroup.

Let $H_4$ be the 4-dimensional Sweedler Hopf algebra. The basis of $H_4$ is $\{1, x, y, w = xy\}$ and the multiplication table is defined by

| | $x$ | $y$ | $w$ |
|---|---|---|---|
| $x$ | $1$ | $w$ | $y$ |
| $y$ | $-w$ | $0$ | $0$ |
| $w$ | $-y$ | $0$ | $0$ |

The costructure of $H_4$ is given by

$$\delta_{H_4}(x) = x \otimes x,\ \delta_{H_4}(y) = y \otimes x + 1 \otimes y,\ \delta_{H_4}(w) = w \otimes 1 + x \otimes w,$$

$$\varepsilon_{H_4}(x) = 1_{\mathbb{F}},\ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0$$

and the antipode $\lambda_{H_4}$ is described by

$$\lambda_{H_4}(x) = x,\ \lambda_{H_4}(y) = w,\ \lambda_{H_4}(w) = -y.$$

The morphism $\tau : \mathbb{F}[L] \otimes H_4 \to \mathbb{F}$ defined by

$$\tau(\sigma_i u^\alpha \otimes z) = \begin{cases} 1 & \text{si} \quad z = 1 \\ (-1)^\alpha & \text{si} \quad z = x \\ 0 & \text{si} \quad z = y, w \end{cases}$$

is a skew pairing. Then, $\mathbb{F}[L] \bowtie_\tau H_4$ is a Hopf quasigroup asociated to the $a$-comonoidal distributive law $\Psi$ of $\mathbb{F}[L]$ over $H_4$ where

$$\Psi(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes 1, \;\; \Psi(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes x,$$

$$\Psi(y \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes y, \;\; \Psi(w \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes w.$$


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The morphism $\tau : \mathbb{F}[L] \otimes H_4 \to \mathbb{F}$ defined by

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$$\Psi(y \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes y, \;\; \Psi(w \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes w.$$

More concretely, $\mathbb{F}[L] \bowtie_\tau H_4 = \mathbb{F}[L] \bowtie H_4$ for the matched pair $(\mathbb{F}[L], H_4)$ where the actions are:

$$\varphi_{\mathbb{F}[L]}(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha, \; \varphi_{\mathbb{F}[L]}(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha, \; \varphi_{\mathbb{F}[L]}(y \otimes \sigma_i u^\alpha) = \varphi_{\mathbb{F}[L]}(w \otimes \sigma_i u^\alpha) = 0$$

and

$$\phi_{H_4}(1 \otimes \sigma_i u^\alpha) = 1, \;\; \phi_{H_4}(x \otimes \sigma_i u^\alpha) = x, \;\; \phi_{H_4}(y \otimes \sigma_i u^\alpha) = (-1)^\alpha y,$$

$$\phi_{H_4}(w \otimes \sigma_i u^\alpha) = (-1)^\alpha w.$$

The multiplication table of $\mathbb{F}[L] \bowtie_{\tau} H_4$ is

| | $\sigma_j u^{\beta} \otimes 1$ | $\sigma_j u^{\beta} \otimes x$ |
|---|---|---|
| $\sigma_j u^{\alpha} \otimes 1$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes 1$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes x$ |
| $\sigma_j u^{\alpha} \otimes x$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes x$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes 1$ |
| $\sigma_j u^{\alpha} \otimes y$ | $(-1)^{\beta} \sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes y$ | $(-1)^{\beta+1} \sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes w$ |
| $\sigma_j u^{\alpha} \otimes w$ | $(-1)^{\beta} \sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes w$ | $(-1)^{\beta+1} \sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes y$ |

| | $\sigma_j u^{\beta} \otimes y$ | $\sigma_j u^{\beta} \otimes w$ |
|---|---|---|
| $\sigma_j u^{\alpha} \otimes 1$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes y$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes w$ |
| $\sigma_j u^{\alpha} \otimes x$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes w$ | $\sigma_i u^{\alpha} \bullet \sigma_j u^{\beta} \otimes y$ |
| $\sigma_j u^{\alpha} \otimes y$ | 0 | 0 |
| $\sigma_j u^{\alpha} \otimes w$ | 0 | 0 |

The antipode is

$$\lambda_{\mathbb{F}[L]\bowtie_{\tau} H_{\mathbf{4}}}(\sigma_i u^{\alpha} \otimes 1) = \sigma_i^{(-\mathbf{1})^{\alpha+\mathbf{1}}} u^{\alpha} \otimes 1,$$

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**Theorem**

$\mathbb{F}[M(S_3, 2)] \bowtie_{\tau} H_4$ is a Hopf quasigroup that is neither commutative nor cocommutative.

# Factorizations of Hopf quasigroups

A Hopf algebra $X$ in $\mathbb{F}$-Vect factorises as

$$X = AH$$

if $A$ and $H$ are sub-Hopf algebras of $X$, with inclusion maps $i_A$ and $i_H$, such that the map

$$\omega(a \otimes h) = i_A(a) i_H(h)$$

is an isomorphism of vector spaces.

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$X$ factorises as $X = AH$ if, and only if, there exists a matched pair of Hopf algebras $(A, H)$ such that $X$ is isomorphic to the double cross product $A \bowtie H$ as Hopf algebras.

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The main objective of this final section is to extend this result to the theory of factorizations of Hopf quasigroups in a symmetric monoidal category $\mathcal{C}$.

## Definition

Let $X$ be a Hopf quasigroup in C. Let $H$, $A$ be Hopf subquasigroups of $X$ with inclusion morphisms $i_H : H \to X$, $i_A : A \to X$ respectively. Let $\omega_X$ and $\theta_X$ be the morphisms defined by

$$\omega_X = \mu_X \circ (i_A \otimes i_H) : A \otimes H \to X, \quad \theta_X = \mu_X \circ (i_H \otimes i_A) : H \otimes A \to X.$$

We will say that $X$ factorizes as

$$X = AH$$

if $\omega_X$ is an isomorphism and the following identities

$$\mu_X \circ (\omega_X \otimes X) = \mu_X \circ (i_A \otimes (\mu_X \circ (i_H \otimes X))),$$

$$\mu_X \circ (X \otimes \omega_X) = \mu_X \circ ((\mu_X \circ (X \otimes i_A)) \otimes i_H),$$

$$\mu_X \circ ((\theta_X \circ (\lambda_H \otimes \lambda_A)) \otimes X) = \mu_X \circ ((i_H \circ \lambda_H) \otimes (\mu_X \circ ((i_A \circ \lambda_A) \otimes X))),$$

$$\mu_X \circ (X \otimes (\theta_X \circ (\lambda_H \otimes \lambda_A))) = \mu_X \circ ((\mu_X \circ (X \otimes (i_H \circ \lambda_H)) \otimes (i_A \circ \lambda_A)),$$

hold.

Then, if the antipodes of $H$ and $A$ are isomorphisms, then we can remove the antipodes in the factorization definition. Then the two last identities become in

$$\mu_X \circ (\theta_X \otimes X) = \mu_X \circ (i_H \otimes (\mu_X \circ (i_A \otimes X)),$$

$$\mu_X \circ (X \otimes \theta_X) = \mu_X \circ ((\mu_X \circ (X \otimes i_H)) \otimes i_A).$$

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### Example

Suppose that $(A, H)$ is a matched pair of Hopf quasigroups. The double cross product

$$A \bowtie H$$

is a Hopf quasigroup.
The morphisms $i_A = A \otimes \eta_H : A \to A \bowtie H$ and $i_H = \eta_A \otimes H : H \to A \bowtie H$ are morphisms of Hopf quasigroups.
Also, we obtain that

$$\omega_{A \bowtie H} = id_{A \otimes H}, \quad \theta_{A \bowtie H} = \Psi$$

and therefore any matched pair of Hopf quasigroups induces an example of factorization.


Wreath products and factorizations for Hopf quasigroups

**Theorem**

Let $H$, $A$ be Hopf subquasigroups of a Hopf quasigroup $X$. If $X$ factorises as $X = AH$, the morphism

$$\Psi = \omega_X^{-1} \circ \theta_X : H \otimes A \to A \otimes H$$

is a comonoidal distributive law of $H$ over $A$. Moreover, if the antipodes of $H$ and $A$ are isomorphisms $\Psi$ is an $a$-comonoidal distributive law of $H$ over $A$.

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**Theorem**

Let $H$, $A$ be Hopf subquasigroups of a Hopf quasigroup $X$ such that the antipodes of $H$ and $A$ are isomorphisms. Assume that $X$ factorises as $X = AH$. Then, $\omega_X$ is an isomorphism of Hopf quasigrous between the wreath product $A \otimes_\Psi H$ and $X$, where $\Psi$ is the $a$-comonoidal distributive law defined in the previous theorem.

**Theorem**

Let $H$, $A$, $X$ be Hopf quasigroups such that the antipodes of $H$ and $A$ are isomorphisms. If $X$ factorizes as $X = AH$, there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.


Wreath products and factorizations for Hopf quasigroups

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### Proof

Let $\Psi$ be the morphism defined in the previous theorems. Define the actions by

$$\varphi_A = (A \otimes \varepsilon_H) \circ \Psi, \quad \phi_H = (\varepsilon_A \otimes H) \circ \Psi.$$

**Theorem**

Let $H$, $A$, $X$ be Hopf quasigroups such that the antipodes of $H$ and $A$ are isomorphisms. Then, $X$ factorizes as $X = AH$ if, and only if, there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.

### Theorem

Let $H$, $A$, $X$ be Hopf quasigroups such that the antipodes of $H$ and $A$ are isomorphisms. Then, $X$ factorizes as $X = AH$ if, and only if, there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.

**References**:

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- **R. González Rodríguez**: Distributive laws in a non-associative setting, (2025) (preprint).

## Theorem

Let $H$, $A$, $X$ be Hopf quasigroups such that the antipodes of $H$ and $A$ are isomorphisms. Then, $X$ factorizes as $X = AH$ if, and only if, there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.

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# Thank you