

Weak non associative objects

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- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez:** Weak Hopf quasigroups, *Asian Journal of Mathematics* 20, N. 4, 665-694 (2016).
- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez:** Cleft and Galois extensions associated to a weak Hopf quasigroup *Journal of Pure and Applied Algebra* 220, No. 3, 1002-1034, (2016).
- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez:** A characterization of weak Hopf (co)quasigroups, *Mediterranean Journal of Mathematics* 13, N. 5, 3747-3764 (2016).
- **J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez** Quasigroupoids and weak Hopf quasigroups, *Journal of Algebra* 568, 408-436 (2021).
- **R. González Rodríguez** Galois extensions for quasigroupoid magmas, *Journal of Algebra and its Applications* 21, No. 11, 2250223 (2022).
- **R. González Rodríguez** Natural transformations for quasigroupoids *Georgian Mathematical Journal* 30(4), 523-538 (2023).
- **R. González Rodríguez** Weak Hopf quasigroups and matched pairs of quasigroupoids *Journal of Algebra and its Applications* (in press) (2026).

Outline

- 1 Preliminaries
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Preliminaries

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- From now on $\mathcal{C}$ denotes a **braided monoidal category** with tensor product denoted by $\otimes$ and unit object K . With c we will denote the braiding.

Recall that a **monoidal category** is a category $\mathcal{C}$ equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a unit object K of $\mathcal{C}$ and a family of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P),$$

$$r_M : M \otimes K \rightarrow M, \quad l_M : K \otimes M \rightarrow M,$$

in $\mathcal{C}$ (called associativity, right unit and left unit constraints, respectively) satisfying the **Pentagon Axiom** and the **Triangle Axiom**, i.e.,

$$a_{M,N,P \otimes Q} \circ a_{M \otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N \otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where id_X denotes the identity morphism for each object X in $\mathcal{C}$.

On the other hand, $\mathcal{C}$ is **braided** if for any couple (M, N) of objects in the category there exists a natural isomorphism

$$c_{M,N} : M \otimes N \rightarrow N \otimes M$$

satisfying the **Hexagon Axiom**, i.e., for all M, N and P in $\mathcal{C}$, the following identities hold:

$$a_{N,P,M} \circ c_{M,N \otimes P} \circ a_{M,N,P} = (id_N \otimes c_{M,P}) \circ a_{N,M,P} \circ (c_{M,N} \otimes id_P),$$

$$a_{P,M,N}^{-1} \circ c_{M \otimes N,P} \circ a_{M,N,P}^{-1} = (c_{M,P} \otimes id_N) \circ a_{M,P,N}^{-1} \circ (id_M \otimes c_{N,P}),$$

- Taking into account that every non-strict monoidal category is monoidal equivalent to a strict one (the constraints are identities), **we can assume without loss of generality that the category is strict** and, as a consequence, the results contained in this talk remain valid for every non-strict braided monoidal category.
- We also assume that every idempotent morphism $q : Y \rightarrow Y$ in $\mathcal{C}$ splits (**$\mathcal{C}$ is Cauchy complete**), i.e. there exist an object $Im(q)$ (**image of q**) and morphisms $i : Im(q) \rightarrow Y$ (**injection**) and $p : Y \rightarrow Im(q)$ (**projection**) such that $q = i \circ p$ and $p \circ i = id_{Im(q)}$.

The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with equalizers or coequalizers.

- For simplicity of notation, given three objects V, U, M in $\mathcal{C}$ and a morphism $f : V \rightarrow U$, we write

$$M \otimes f \text{ for } id_M \otimes f \text{ and } f \otimes M \text{ for } f \otimes id_M.$$

- The triple

$$(A, \eta_A, \mu_A)$$

is a **unital magma**, if $\eta_A : K \rightarrow A$ (unit) and $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in $\mathcal{C}$ such that

$$\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A).$$

If (A, η_A, μ_A) is a unital magma and

$$\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$$

we will say that A is an **algebra**.

Let (A, η_A, μ_A) and (B, η_B, μ_B) be unital magmas (algebras). A morphism

$$f : A \rightarrow B$$

in $\mathcal{C}$ is a **morphism of unital magmas** if

$$\eta_B = f \circ \eta_A, \quad f \circ \mu_A = \mu_B \circ (f \otimes f).$$

- The triple

$$(C, \varepsilon_C, \delta_C)$$

is a **counital comagma** if $\varepsilon_C : C \rightarrow K$ (counit) and $\delta_C : C \rightarrow C \otimes C$ (comultiplication) are morphisms in C such that

$$(\varepsilon_C \otimes C) \circ \delta_C = id_C = (C \otimes \varepsilon_C) \circ \delta_C$$

If moreover, the following identity holds

$$(\delta_C \otimes C) \circ \delta_C = (C \otimes \delta_C) \circ \delta_C$$

we will say that $(C, \varepsilon_C, \delta_C)$ is a **coalgebra**.

Let $(C, \varepsilon_C, \delta_C)$ and $(D, \varepsilon_D, \delta_D)$ be counital comagmas (coalgebras). A morphism

$$f : C \rightarrow D$$

in C is a **morphism of counital comagmas** if

$$\varepsilon_C = \varepsilon_D \circ f, \quad \delta_D \circ f = (f \otimes f) \circ \delta_C.$$

Let $(D, \varepsilon_D, \delta_D)$ be a counital comagma (coalgebra). We will say that D is **cocommutative** if

$$\delta_D = c_{D,D} \circ \delta_D.$$

- If $f, g : C \rightarrow A$ are morphisms between a counital comagma C and a unital magma A ,

$$f * g$$

denotes the **convolution product** defined by

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_C.$$

Weak Hopf quasigroups

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Definition

A **weak non associative bialgebra** H in $\mathbf{C}$ is a unital magma (H, η_H, μ_H) and a coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H).$
- (2)
$$\begin{aligned} \varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) &= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H) \\ &= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H}^{-1} \circ \delta_H) \otimes H). \end{aligned}$$
- (3)
$$\begin{aligned} (\delta_H \otimes H) \circ \delta_H \circ \eta_H &= (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)) \\ &= (H \otimes (\mu_H \circ c_{H,H}^{-1}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H)). \end{aligned}$$

- If the product μ_H is associative, the previous notion is the one of weak bialgebra in a braided monoidal category.
- If the product μ_H is associative and ε_H and δ_H are algebra morphisms, we have the notion of bialgebra in a braided monoidal category.
- If the ε_H and δ_H are morphisms of unital magmas, we have the notion of non associative bialgebra in a braided monoidal category.

Theorem

Let H be a weak non associative bialgebra. The morphisms

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \rightarrow H,$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \rightarrow H,$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H) : H \rightarrow H,$$

and

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \rightarrow H$$

are idempotent.

Notation

Let H be a weak non associative bialgebra. The morphism Π_H^L will be called **target morphism** and Π_H^R will be called **source morphism**.

Theorem

Let H be a weak non associative bialgebra. The following identities hold:

$$\pi_H^L \circ \bar{\pi}_H^L = \pi_H^L, \quad \pi_H^L \circ \bar{\pi}_H^R = \bar{\pi}_H^R, \quad \bar{\pi}_H^L \circ \pi_H^L = \bar{\pi}_H^L, \quad \bar{\pi}_H^R \circ \pi_H^L = \pi_H^L,$$

$$\pi_H^R \circ \bar{\pi}_H^L = \bar{\pi}_H^L, \quad \pi_H^R \circ \bar{\pi}_H^R = \pi_H^R, \quad \bar{\pi}_H^L \circ \pi_H^R = \pi_H^R, \quad \bar{\pi}_H^R \circ \pi_H^R = \bar{\pi}_H^R.$$

Remark

Let H be a weak non associative bialgebra. Any equality containing the idempotent morphisms Π_H^L , Π_H^R , $\overline{\Pi}_H^L$, $\overline{\Pi}_H^R$, the (co)product and the (co)unit, that holds for weak bialgebras also holds for H .

For example:

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H),$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H),$$

$$\mu_H \circ (H \otimes \bar{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H),$$

$$\mu_H \circ (\bar{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H),$$

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$(\bar{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$(H \otimes \bar{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$\Pi_H^L \circ \mu_H \circ (H \otimes \Pi_H^L) = \Pi_H^L \circ \mu_H,$$

$$\Pi_H^R \circ \mu_H \circ (\Pi_H^R \otimes H) = \Pi_H^R \circ \mu_H,$$

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L,$$

$$(\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R,$$

Theorem

Let H be a weak non associative bialgebra. Put $H_L = \text{Im}(\Pi_H^L)$ and let $p_L : H \rightarrow H_L$ and $i_L : H_L \rightarrow H$ be the morphisms such that $\Pi_H^L = i_L \circ p_L$ and $p_L \circ i_L = \text{id}_{H_L}$. Then,

$$\begin{array}{ccccc}
 H_L & \xrightarrow{i_L} & H & \begin{array}{c} \xrightarrow{\delta_H} \\ \xrightarrow{(H \otimes \Pi_H^L) \circ \delta_H} \end{array} & H \otimes H
 \end{array}$$

is an equalizer diagram and

$$\begin{array}{ccccc}
 H \otimes H & \begin{array}{c} \xrightarrow{\mu_H} \\ \xrightarrow{\mu_H \circ (H \otimes \Pi_H^L)} \end{array} & H & \xrightarrow{p_L} & H_L
 \end{array}$$

is a coequalizer diagram.

As a consequence, $(H_L, \eta_{H_L} = p_L \circ \eta_H, \mu_{H_L} = p_L \circ \mu_H \circ (i_L \otimes i_L))$ is a unital magma in $\mathcal{C}$. Also, $(H_L, \varepsilon_{H_L} = \varepsilon_H \circ i_L, \delta_H = (p_L \otimes p_L) \circ \delta_H \circ i_L)$ is a coalgebra in $\mathcal{C}$.

Theorem

Let H be a weak non associative bialgebra. The following identities hold:

$$\mu_H \circ ((\mu_H \circ (i_L \otimes H)) \otimes H) = \mu_H \circ (i_L \otimes \mu_H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (i_L \otimes H))) = \mu_H \circ ((\mu_H \circ (H \otimes i_L)) \otimes H),$$

$$\mu_H \circ (H \otimes (\mu_H \circ (H \otimes i_L))) = \mu_H \circ (\mu_H \otimes i_L).$$

As a consequence, the unital magma H_L is a algebra in $\mathcal{C}$.

If $H_R = \text{Im}(\Pi_H^R)$ we have the same properties and then H_R is a algebra in $\mathcal{C}$.

Definition

A **weak Hopf quasigroup** H in $\mathcal{C}$ is a weak non associative bialgebra for which exists an endomorphism $\lambda_H : H \rightarrow H$ in $\mathcal{C}$ (called the **antipode** of H) such that the following equalities hold:

- (1) $\Pi_H^L = id_H * \lambda_H.$
- (2) $\Pi_H^R = \lambda_H * id_H.$
- (3) $\lambda_H * \Pi_H^L = \Pi_H^R * \lambda_H = \lambda_H.$
- (4) $\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^R \otimes H).$
- (5) $\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) = \mu_H \circ (\Pi_H^L \otimes H).$
- (6) $\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^L).$
- (7) $\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = \mu_H \circ (H \otimes \Pi_H^R).$

- If the product μ_H is associative, the previous notion is the one of **weak braided Hopf algebra** introduced in
J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez, Weak braided Hopf algebras, **Indiana University Mathematics Journal** 57, No. 5, 2423-2458 (2008).
- If the product μ_H is associative and $C = \mathbb{K}\text{Vect}$, weak Hopf quasigroups are **weak Hopf algebras** as was defined by:
G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras, I. Integral theory and C^* -structure, **J. Algebra** 221, 385-438 (1999).
- If the product μ_H is associative and ε_H and δ_H are algebra morphisms, we have the notion of **Hopf algebra** in a braided monoidal category.
- If ε_H and δ_H are morphisms of unital magmas and $C = \mathbb{K}\text{Vect}$, we get the notion of **Hopf quasigroup** defined in:
J. Klim, S. Majid, Hopf quasigroups and the algebraic 7-sphere, **J. Algebra** 323, 3067-3110 (2010).

Definition

A **Hopf quasigroup** H in $\mathbf{C}$ is a unital magma (H, η_H, μ_H) and a coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (1) The counit ε_H and the coproduct δ_H are morphisms of unital magmas.
- (2) There exists $\lambda_H : H \rightarrow H$ in $\mathbf{C}$ (called the antipode of H) such that:

$$(2-1) \quad \mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H).$$

$$(2-2) \quad \mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H).$$

Example

Suppose that $\mathbf{C} = \mathbf{Set}$. Then L is a Hopf quasigroup in $\mathbf{C}$ if, and only if, L is an **I.P. loop**.

An I.P. loop is a set L with a product, identity e_L , and with the property that for each $u \in L$ there exists $u^{-1} \in L$ (the inverse of u) such that

$$u^{-1}(uv) = v, \quad (vu)u^{-1} = v, \quad \forall v \in L.$$

Note that in this case L is a cocommutative Hopf quasigroup because $\delta_L(u) = (u, u)$.

Theorem

The antipode of a weak Hopf quasigroup H is unique and leaves the unit and the counit invariant, i.e. $\lambda_H \circ \eta_H = \eta_H$ and $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

Theorem

Let H be a weak Hopf quasigroup. The following identities hold:

$$\Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R, \quad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L,$$

$$\Pi_H^L = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R.$$

Theorem

Let H be a weak Hopf quasigroup. The antipode of H is antimultiplicative and anticomultiplicative, i.e. the following equalities hold:

$$\lambda_H \circ \mu_H = \mu_H \circ c_{H,H} \circ (\lambda_H \otimes \lambda_H),$$

$$\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H,$$

Definition

Let H, H' be weak Hopf quasigroups. We will say that $f : H \rightarrow H'$ is a **morphism of weak Hopf quasigroups** if it is a coalgebra morphism such that

$$\begin{aligned}\Pi_{H'}^R \circ f &= f \circ \Pi_H^R, \\ \overline{\Pi}_{H'}^L \circ f &= f \circ \overline{\Pi}_H^L, \\ \Pi_{H'}^R \circ \Pi_{H'}^L \circ f &= f \circ \Pi_H^R \circ \Pi_H^L, \\ f \circ \mu_H &= \mu_{H'} \circ (f \otimes f) \circ \nabla_H,\end{aligned}$$

hold, where $\nabla_H : H \otimes H \rightarrow H \otimes H$ is the idempotent morphism defined by

$$\nabla_H = (H \otimes (\mu_H \circ (\Pi_H^R \otimes H))) \circ (\delta_H \otimes H).$$

- The previous definition is similar to the one introduced in **G. Böhm, J. Gómez-Torrecillas, E. López Centella**, On the category of weak bialgebras, **J. Algebra** 399 (2014), 801-844.

for weak Hopf algebras.

With WHQ we will denote the category of weak Hopf quasigroups.

Therefore, excepting associativity, weak Hopf quasigroups satisfy the relevant equalities that we can find in the theory of weak Hopf algebras (or quantum groupoids following Nikshych and Vainerman).

D. Nikshych, L. Vainerman, Finite quantum groupoids and their applications, *New directions in Hopf algebras*. Cambridge: Cambridge University Press. Math. Sci. Res. Inst. Publ. 43 (2002), 211-262.

Galois morphisms and the characterization of weak Hopf quasigroups

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Let H be an algebra-coalgebra in $\mathcal{C}$.

Definition

- (i) The **right Galois morphism** $\beta : H \otimes H \rightarrow H \otimes H$ is defined as $\beta = (\mu_H \otimes H) \circ (H \otimes \delta_H)$.
- (ii) The **left Galois morphism** $\gamma : H \otimes H \rightarrow H \otimes H$ is defined as $\gamma = (H \otimes \mu_H) \circ (\delta_H \otimes H)$.

Theorem

Let H be a bialgebra. The following assertions are equivalent.

- (i) H is a Hopf algebra.
- (ii) The right Galois morphism is an isomorphism.
- (iii) The left Galois morphism is an isomorphism.

Proof

- (i) $\Rightarrow$ (ii) $\beta^{-1} = (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H)$
- (ii) $\Rightarrow$ (i) $\lambda_H = (H \otimes \varepsilon_H) \circ \beta^{-1} \circ (\eta_H \otimes H)$

Now let H be a weak non associative bialgebra. Define the Ω -morphisms,

$$\Omega_L^1 = (\mu_H \otimes H) \circ (H \otimes \Pi_H^L \otimes H) \circ (H \otimes \delta_H),$$

$$\Omega_R^1 = (\mu_H \otimes H) \circ (H \otimes \Pi_H^R \otimes H) \circ (H \otimes \delta_H),$$

$$\Omega_L^2 = (H \otimes \mu_H) \circ (H \otimes \Pi_H^L \otimes H) \circ (\delta_H \otimes H),$$

$$\Omega_R^2 = (H \otimes \mu_H) \circ (H \otimes \Pi_H^R \otimes H) \circ (\delta_H \otimes H).$$

These morphisms are idempotent and, as a consequence, there exist objects $H \times_L^1 H$, $H \times_R^1 H$, $H \times_L^2 H$ and $H \times_R^2 H$ and morphisms

$$p_L^1 : H \otimes H \rightarrow H \times_L^1 H, \quad i_L^1 : H \times_L^1 H \rightarrow H \otimes H,$$

$$p_R^1 : H \otimes H \rightarrow H \times_R^1 H, \quad i_R^1 : H \times_R^1 H \rightarrow H \otimes H,$$

$$p_L^2 : H \otimes H \rightarrow H \times_L^2 H, \quad i_L^2 : H \times_L^2 H \rightarrow H \otimes H,$$

$$p_R^2 : H \otimes H \rightarrow H \times_R^2 H, \quad i_R^2 : H \times_R^2 H \rightarrow H \otimes H,$$

such that, for $\sigma \in \{L, R\}$ and $\alpha \in \{1, 2\}$,

$$i_\sigma^\alpha \circ p_\sigma^\alpha = \Omega_\sigma^\alpha, \quad p_\sigma^\alpha \circ i_\sigma^\alpha = id_{H \times_\sigma^\alpha H}, \quad H \times_\sigma^\alpha H = Im(\Omega_\sigma^\alpha).$$

Theorem

Let H be a weak non associative bialgebra. Then we have that:

(i) The diagrams

$$\begin{array}{ccc} H \otimes H_L \otimes H & \xrightarrow{\begin{array}{c} (\mu_H \circ (H \otimes i_L)) \otimes H \\ H \otimes (\mu_H \circ (i_L \otimes H)) \end{array}} & H \otimes H \xrightarrow{p_L^1} H \times_L^1 H \end{array}$$

and

$$\begin{array}{ccc} H \otimes H_R \otimes H & \xrightarrow{\begin{array}{c} (\mu_H \circ (H \otimes i_R)) \otimes H \\ H \otimes (\mu_H \circ (i_R \otimes H)) \end{array}} & H \otimes H \xrightarrow{p_R^2} H \times_R^2 H \end{array}$$

are coequalizer diagrams.

(ii) The diagrams

$$\begin{array}{ccc} H \times_L^2 H & \xrightarrow{i_L^2} H \otimes H \xrightarrow{\begin{array}{c} ((H \otimes p_L) \circ \delta_H) \otimes H \\ H \otimes ((p_L \otimes H) \circ \delta_H) \end{array}} & H \otimes H_L \otimes H \end{array}$$

and

$$\begin{array}{ccc} H \times_R^1 H & \xrightarrow{i_R^1} H \otimes H \xrightarrow{\begin{array}{c} ((H \otimes p_R) \circ \delta_H) \otimes H \\ H \otimes ((p_R \otimes H) \circ \delta_H) \end{array}} & H \otimes H_R \otimes H \end{array}$$

are equalizer diagrams.

Definition

Let H be a unital magma. We say that a morphism $\phi : H \otimes H \rightarrow H \otimes H$ is:

(i) **Almost left H -linear**, if $\phi = (\mu_H \otimes H) \circ (H \otimes \phi) \circ (H \otimes \eta_H \otimes H)$.

(ii) **Almost right H -linear**, if $\phi = (H \otimes \mu_H) \circ (\phi \otimes H) \circ (H \otimes \eta_H \otimes H)$.

By dualization, if H is a counital comagma, we will say that a morphism ϕ is **almost left H -colinear** if

$$\phi = (H \otimes \varepsilon_H \otimes H) \circ (H \otimes \phi) \circ (\delta_H \otimes H),$$

and **almost right H -colinear** if

$$\phi = (H \otimes \varepsilon_H \otimes H) \circ (\phi \otimes H) \circ (H \otimes \delta_H).$$

Theorem

Let H be a unital magma and a counital comagma. The following assertions hold.

- (i) The right Galois morphism $\beta = (\mu_H \otimes H) \circ (H \otimes \delta_H)$ is almost left H -linear and almost right H -colinear.
- (ii) The left Galois morphism $\gamma = (H \otimes \mu_H) \circ (\delta_H \otimes H)$ is almost right H -linear and almost left H -colinear.
- (iii) The morphisms Ω_L^1 and Ω_R^1 are almost left H -linear and almost right H -colinear.
- (iv) The morphisms Ω_L^2 and Ω_R^2 are almost right H -linear and almost left H -colinear.

Note that, if H is a weak Hopf quasigroup, we can express the Ω -morphisms as compositions of the Galois morphisms in the following way

$$\Omega_L^1 = \bar{\beta} \circ \beta, \quad \Omega_R^1 = \beta \circ \bar{\beta}, \quad \Omega_L^2 = \gamma \circ \bar{\gamma}, \quad \Omega_R^2 = \bar{\gamma} \circ \gamma,$$

where

$$\bar{\beta} = (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H)$$

and

$$\bar{\gamma} = (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H).$$

Moreover, if the weak Hopf quasigroup H is a Hopf quasigroup,

$$\Pi_H^L = \Pi_H^R = \bar{\Pi}_H^L = \bar{\Pi}_H^R = \varepsilon_H \otimes \eta_H$$

and then the Ω -morphisms are identities. As a consequence we have that in this case the Galois morphisms β and γ are isomorphisms with inverses $\bar{\beta}$ and $\bar{\gamma}$, respectively.

Theorem

Let H be a weak non associative bialgebra. The following assertions are equivalent:

- (i) H is a weak Hopf quasigroup.
- (ii) The morphisms

$$f = p_R^1 \circ \beta \circ i_L^1 : H \times_L^1 H \rightarrow H \times_R^1 H$$

and

$$g = p_L^2 \circ \gamma \circ i_R^2 : H \times_R^2 H \rightarrow H \times_L^2 H$$

are isomorphisms, the morphism $i_L^1 \circ f^{-1} \circ p_R^1$ is almost left H -linear and $i_R^2 \circ g^{-1} \circ p_L^2$ is almost right H -linear.

Proof

- (i) $\Rightarrow$ (ii) $f^{-1} = p_L^1 \circ \bar{\beta} \circ i_R^1$ and $g^{-1} = p_R^2 \circ \bar{\gamma} \circ i_L^2$
- (ii) $\Rightarrow$ (i) $\lambda_H = (H \otimes \varepsilon_H) \circ i_R^1 \circ f^{-1} \circ p_R^1 \circ (\eta_H \otimes H)$

Corollary

Let H be a unital magma and coalgebra such that ε_H and δ_H are morphisms of unital magmas. Then H is a Hopf quasigroup if and only if the Galois morphisms β and γ are isomorphisms and they have almost left H -linear and almost right H -linear inverses, respectively.

- This result was proved by T. Brzezinski for Hopf quasigroups in $\mathbb{K}\text{Vect}$ in:
T. Brzeziński, Hopf modules and the fundamental theorem for Hopf quasigroups,
Internat. Elec. J. Algebra 8, 114-128 (2010).

Corollary

Let H be a weak bialgebra. The following assertions are equivalent.

(i) H is a weak Hopf algebra.

(ii) The morphism

$$f = p_R^1 \circ \beta \circ i_L^1 : H \times_L^1 H \rightarrow H \times_R^1 H$$

is an isomorphism.

(iii) The morphism

$$g = p_L^2 \circ \gamma \circ i_R^2 : H \times_R^2 H \rightarrow H \times_L^2 H$$

is an isomorphism.

- This result was proved by P. Schauenburg for weak Hopf algebras in $\mathbb{K}\text{Vect}$ in:
P. Schauenburg, Weak Hopf algebras and quantum groupoids, **Noncommutative geometry and quantum groups (Warsaw, 2001)**, 171-188, Polish Acad. Sci., Warsaw, (2003).

Quasigroupoids

- 1 Preliminaries
- 2 Weak Hopf quasigroups
- 3 Galois morphisms and the characterization of weak Hopf quasigroups
- 4 Quasigroupoids**
- 5 Categorical equivalences
- 6 Matched pairs of quasigroupoids

Notation

Let X , Y , Z be sets and

$$\alpha : X \rightarrow Z, \quad \beta : Y \rightarrow Z$$

be maps.

Then

$$X \alpha \times \beta Y$$

denotes the pullback of α and β , i.e.,

$$X \alpha \times \beta Y = \{(x, y) \in X \times Y ; \alpha(x) = \beta(y)\}$$

Definition

A **quasigroupoid** (I.P. Loopoid) A is an ordered pair of sets $A = (A_0, A_1)$ such that:

- (1) There exist maps $s_A : A_1 \rightarrow A_0$, $t_A : A_1 \rightarrow A_0$, and $id_A : A_0 \rightarrow A_1$, called source, target and identity, respectively, satisfying

$$s_A(id_A(x)) = t_A(id_A(x)) = x, \quad \forall x \in A_0.$$

- (2) There exist a map, called product of A ,

$$\bullet : A_1 \times_{s_A, t_A} A_1 \rightarrow A_1,$$

defined by $\bullet(a, b) = a \bullet b$ and a map $\lambda_A : A_1 \rightarrow A_1$, called the inverse map, such that:

- (2-1) For each $a \in A_1$,

$$id_A(t_A(a)) \bullet a = a = a \bullet id_A(s_A(a)).$$

- (2-2) For all $(a, b) \in A_1 \times_{s_A, t_A} A_1$,

$$s_A(a \bullet b) = s_A(b), \quad t_A(a \bullet b) = t_A(a).$$

- (2-3) For all $(a, b) \in A_1 \times_{s_A, t_A} A_1$, $(\lambda_A(a), a \bullet b)$ and $(a \bullet b, \lambda_A(b))$ are in $A_1 \times_{s_A, t_A} A_1$ and

$$\lambda_A(a) \bullet (a \bullet b) = b, \quad (a \bullet b) \bullet \lambda_A(b) = a.$$

- The set A_0 will be called the **base set**, or also the **set of objects**, of A . The set A_1 will be called the **set of arrows** of A . We will say that a quasigroupoid A is **finite** if its base set is finite.
- Note that a finite quasigroupoid where $|A_0| = 1$ is an I.P. loop.
- If the product of a quasigroupoid A is associative, A is a groupoid, i.e., a category where every morphism is an isomorphism.
- A quasigroupoid is an **inverse semiloopoid satisfying the unities associativity assumption**. These notions were introduced by J. Grabowski in
J. Grabowski. An introduction to loopoids, **Comment. Math. Univ. Carolin.** 57 (2016), 515-526.

Theorem

If A is a quasigroupoid, the following equalities

$$s_A(\lambda_A(a)) = t_A(a),$$

$$t_A(\lambda_A(a)) = s_A(a),$$

$$\lambda_A(a) \bullet a = id_A(s_A(a)),$$

$$a \bullet \lambda_A(a) = id_A(t_A(a)),$$

$$\lambda_A(\lambda_A(a)) = a,$$

$$\lambda_A(a \bullet b) = \lambda_A(b) \bullet \lambda_A(a),$$

hold for all $a \in A_1$ and $(a, b) \in A_1 \times_{s_A} A_1$.

Definition

Let A, A' be quasigroupoids. A **quasigroupoid morphism** $\Gamma : A \rightarrow A'$ is a pair of maps $\Gamma = (\Gamma_0, \Gamma_1)$,

$$\Gamma_0 : A_0 \rightarrow A'_0, \quad \Gamma_1 : A_1 \rightarrow A'_1,$$

such that

- (1) $\Gamma_0 \circ s_A = s_{A'} \circ \Gamma_1$,
- (2) $\Gamma_0 \circ t_A = t_{A'} \circ \Gamma_1$,
- (3) For all $x \in A_0$, $\Gamma_1(id_A(x)) = id_{A'}(\Gamma_0(x))$,
- (4) For all $(a, b) \in A_1 \times_{s_A} A_1$, $\Gamma_1(a \bullet b) = \Gamma_1(a) \bullet' \Gamma_1(b)$.

The obvious composition of quasigroupoid morphisms is a quasigroupoid morphism. Then with QGPD we will denote the category whose objects are quasigroupoids and whose morphisms are morphisms of quasigroupoids. With $\overline{\text{QGPD}}$ we will denote the full subcategory of QGPD whose objects are finite quasigroupoids.

Example

Let A be an I.P. loop with product $\cdot$ and let X be a set. Assume that there exists a map

$$\psi_X : A \times X \rightarrow X$$

satisfying the following two conditions:

$$\psi_X(e_A, x) = x,$$

$$\psi_X(a \cdot b, x) = \psi_X(a, \psi_X(b, x)),$$

for all $x \in X$ and $a, b \in A$ (in this case we will say that ψ_X is an action of A on X).

The quasigroupoid $B = (B_0, B_1)$ associated to the action ψ_X is defined by the sets $B_0 = X$, $B_1 = A \times X$ and maps

$$\begin{aligned}s_B : B_1 &\rightarrow B_0, & s_B(a, x) &= x, \\ t_B : B_1 &\rightarrow B_0, & t_B(a, x) &= \psi_X(a, x), \\ id_B : B_0 &\rightarrow B_1, & id_B(x) &= (e_A, x).\end{aligned}$$

Then,

$$B_1 \times_{s_B \times t_B} B_1 = \{((a, x), (b, y)) \in B_1 \times B_1 \mid \psi_X(b, y) = x\}$$

and the product is defined by

$$(a, x) \star (b, y) = (a \cdot b, y).$$

The inverse map $\lambda_B : B_1 \rightarrow B_1$ is $\lambda_B(a, x) = (a^{-1}, \psi_X(a, x))$.

- As was proved in

J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez Quasigroupoids and weak Hopf quasigroups **Journal of Algebra** 568, 408-436 (2021)

examples of this kind can be obtained by working with Moufang loops of small order and the 4-dimensional Taft Hopf algebra.

Example

Let A be a I.P. loop and let X be a set. Denote by T the set

$$T = \{(a, x, y) \mid a \in A, x, y \in X\}.$$

The quasigroupoid $T = (T_0, T_1)$ associated to T is defined by the sets

$$T_0 = \{(e_A, x, x) \in T\}, \quad T_1 = T$$

and maps

$$s_T : T_1 \rightarrow T_0, \quad s_T(a, x, y) = (e_A, y, y),$$

$$t_T : T_1 \rightarrow T_0, \quad t_T(a, x, y) = (e_A, x, x),$$

$$id_T : T_0 \rightarrow T_1, \quad id_T(e_A, x, x) = (e_A, x, x).$$

Then,

$$T_1 \times_{s_T \times t_T} T_1 = \{((a, x, y), (b, y, r)) \in T_1 \times T_1\}$$

and the product is defined by

$$(a, x, y) \star (b, y, r) = (a \cdot b, x, r).$$

The inverse map $\lambda_T : T_1 \rightarrow T_1$ is $\lambda_T(a, x, y) = (a^{-1}, y, x)$.

Example

Let A be a quasigroupoid, let P be a set and let $\pi : P \rightarrow A_0$ be a surjective map. The quasigroupoid $P(A)^\pi = (P(A)_0^\pi, P(A)_1^\pi)$ is defined by the sets $P(A)_0^\pi = P$,

$$P(A)_1^\pi = \{(p, a, q) \mid (p, a, q) \in P \times A_1 \times P, \pi(p) = t_A(a), \pi(q) = s_A(a)\}$$

and maps

$$s_{P(A)^\pi} : P(A)_1^\pi \rightarrow P(A)_0^\pi, \quad s_{P(A)^\pi}(p, a, q) = q,$$

$$t_{P(A)^\pi} : P(A)_1^\pi \rightarrow P(A)_0^\pi, \quad t_{P(A)^\pi}(p, a, q) = p,$$

$$id_{P(A)^\pi} : P(A)_0^\pi \rightarrow P(A)_1^\pi, \quad id_{P(A)^\pi}(p, a, q) = (p, id_{\pi(p)}, p).$$

Then,

$$P(A)_1^\pi \times_{s_{P(A)^\pi} \times t_{P(A)^\pi}} P(A)_1^\pi = \{((p, a, q), (q, b, n)) \in P(A)_1^\pi \times P(A)_1^\pi\}$$

and the product is defined by

$$(p, a, q) \star (q, b, n) = (p, a \bullet b, n).$$

The inverse map $\lambda_{P(A)^\pi} : P(A)_1^\pi \rightarrow P(A)_1^\pi$ is

$$\lambda_{P(A)^\pi}(p, a, q) = (q, \lambda_A(a), p).$$

Theorem

Let $\mathbb{K}$ be a field and let $A = (A_0, A_1)$ be a finite quasigroupoid. The **quasigroupoid magma** $\mathbb{K}[A]$, defined by the free vector space

$$\mathbb{K}[A] = \bigoplus_{a \in A_1} \mathbb{K}a,$$

is a cocommutative weak Hopf quasigroup with unit $1 = \sum_{x \in A_0} id_A(x)$, product

$$\mu_{\mathbb{K}[A]}(a \otimes b) = \begin{cases} a \bullet b, & \text{if } (a, b) \in A_1 \times_{s_A} A_1, \\ 0, & \text{if } (a, b) \notin A_1 \times_{s_A} A_1, \end{cases}$$

counit $\varepsilon_{\mathbb{K}[A]}(a) = 1_{\mathbb{K}}$, coproduct $\delta_{\mathbb{K}[A]}(a) = a \otimes a$ and antipode $\lambda_{\mathbb{K}[A]}(a) = \lambda_A(a)$ on the basis elements.

Theorem

Let $\mathbb{K}$ be a field and let $\Gamma = (\Gamma_0, \Gamma_1) : A \rightarrow A'$ be a morphism of finite quasigroupoids. Then the linear extension

$$\mathbb{K}[\Gamma_1] : \mathbb{K}[A] \rightarrow \mathbb{K}[A'], \quad \mathbb{K}[\Gamma_1](a) = \Gamma_1(a),$$

is a morphism of weak Hopf quasigroups that we will denote by $\mathbb{K}[\Gamma]$.

Theorem

There exists a functor, called the **WHQ-functor**,

$$F : \overline{\text{QGPD}} \rightarrow \text{WHQ}$$

defined on objects by $F(A) = \mathbb{K}[A]$ and on morphisms by $F(\Gamma) = \mathbb{K}[\Gamma]$.

Categorical equivalences

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In this section we will assume that $\mathcal{C} = {}_{\mathbb{K}}\mathbf{Vect}$.

If S is a set, with $\mathbb{K}[S]$ we will denote the free $\mathbb{K}$ -vector space on S , i.e.,

$$\mathbb{K}[S] = \bigoplus_{s \in S} \mathbb{K}s.$$

This vector space, called the **grouplike coalgebra**, has a coalgebra structure determined by

$$\epsilon_{\mathbb{K}[S]}(s) = 1_{\mathbb{F}}, \quad \delta_{\mathbb{K}[S]}(s) = s \otimes s.$$

Definition

Let $D = (D, \varepsilon_D, \delta_D)$ be a coalgebra in $\mathcal{C}$. We will say $g \in D$ is a **grouplike element** if

$$\delta_D(g) = g \otimes g$$

and

$$\varepsilon_D(g) = 1.$$

We will denote by $G(D)$ the set of grouplike elements of D . The set $G(D)$ is a subcoalgebra of D .

If S is a set, the coalgebra $\mathbb{K}[S]$ satisfies

$$G(\mathbb{K}[S]) = S.$$

Theorem

Let H be a weak Hopf quasigroup and consider $t \in H$ such that $\delta_H(t) = t \otimes t$. Then the following equalities hold:

- (i) $\Pi_H^L(t)t = \bar{\Pi}_H^L(t)t = t\Pi_H^R(t) = t\bar{\Pi}_H^R(t) = t$.
- (ii) The elements $\Pi_H^L(t)$, $\bar{\Pi}_H^L(t)$, $\Pi_H^R(t)$ and $\bar{\Pi}_H^R(t)$ are idempotent.
- (iii) $\delta_H(\Pi_H^L(t)) = \Pi_H^L(t) \otimes \Pi_H^L(t)$, $\delta_H(\Pi_H^R(t)) = \Pi_H^R(t) \otimes \Pi_H^R(t)$.
- (iv) $(\Pi_H^L \circ \Pi_H^R)(t) = \Pi_H^R(t)$, $(\Pi_H^R \circ \Pi_H^L)(t) = \Pi_H^L(t)$.
- (v) $\Pi_H^L(t) = \bar{\Pi}_H^L(t)$, $\Pi_H^R(t) = \bar{\Pi}_H^R(t)$.
- (vi) $\delta_H(\bar{\Pi}_H^L(t)) = \bar{\Pi}_H^L(t) \otimes \bar{\Pi}_H^L(t)$, $\delta_H(\bar{\Pi}_H^R(t)) = \bar{\Pi}_H^R(t) \otimes \bar{\Pi}_H^R(t)$.
- (vii) $(\Pi_H^L \circ \lambda_H)(t) = \Pi_H^R(t) = (\lambda_H \circ \Pi_H^R)(t)$, $(\Pi_H^R \circ \lambda_H)(t) = \Pi_H^L(t) = (\lambda_H \circ \Pi_H^L)(t)$.
- (viii) $(\lambda_H^2 \circ \Pi_H^L)(t) = \Pi_H^L(t) = (\Pi_H^L \circ \lambda_H^2)(t)$, $(\lambda_H^2 \circ \Pi_H^R)(t) = \Pi_H^R(t) = (\Pi_H^R \circ \lambda_H^2)(t)$.
- (ix) $\lambda_H^2(t) = t$.

Corollary

Let H be a weak Hopf quasigroup. If $g \in H$ is a grouplike element of H , then so are $\Pi_H^L(g)$, $\Pi_H^R(g)$ and $\lambda_H(g)$.

Theorem

Let H be a weak Hopf quasigroup in $\mathcal{C}$. The algebras H_L and H_R are separable, finite dimensional and semisimple.

Theorem

Let H be a weak Hopf quasigroup. The group-like elements of H are linearly independent. Therefore, the cardinality of

$$G(H) \cap H_R$$

is finite.

Theorem

Let H be a weak Hopf quasigroup. The ordered pair of sets $\mathcal{T}(H) = (\mathcal{T}(H)_0, \mathcal{T}(H)_1)$, where

$$\mathcal{T}(H)_0 = G(H) \cap H_R, \quad \mathcal{T}(H)_1 = G(H),$$

is a finite quasigroupoid where the source, target and identity morphisms

$$s : \mathcal{T}(H)_1 \rightarrow \mathcal{T}(H)_0, \quad t : \mathcal{T}(H)_1 \rightarrow \mathcal{T}(H)_0, \quad i\partial : \mathcal{T}(H)_0 \rightarrow \mathcal{T}(H)_1,$$

are defined by

$$s(g) = \Pi_H^R(g), \quad t(g) = \Pi_H^L(g), \quad i\partial(r) = r,$$

the product

$$\star : \mathcal{T}(H)_1 \times_{s \times t} \mathcal{T}(H)_1 \rightarrow \mathcal{T}(H)_1$$

is defined by $h \star g = \mu_H(h \otimes g)$, and the inverse map $\lambda : \mathcal{T}(H)_1 \rightarrow \mathcal{T}(H)_1$ is

$$\lambda(g) = \lambda_H(g).$$

Theorem

Let $f : H \rightarrow H'$ be a morphism of weak Hopf quasigroups. The pair

$$\mathcal{T}(f) = (\mathcal{T}(f)_0, \mathcal{T}(f)_1),$$

where

$$\mathcal{T}(f)_0 : \mathcal{T}(H)_0 \rightarrow \mathcal{T}(H')_0, \quad \mathcal{T}(f)_1 : \mathcal{T}(H)_1 \rightarrow \mathcal{T}(H')_1$$

are the maps defined by $\mathcal{T}(f)_0(r) = f(r)$ and $\mathcal{T}(f)_1(g) = f(g)$, is a morphism of quasigroupoids between $\mathcal{T}(H)$ and $\mathcal{T}(H')$.

Theorem

There exists a functor, called the **QGPD-functor**,

$$L : \text{WHQ} \rightarrow \overline{\text{QGPD}}$$

defined on objects by $L(H) = \mathcal{T}(H)$ and on morphisms by $L(f) = \mathcal{T}(f)$.

Theorem

The WHQ-functor is left adjoint of the QGPD-functor.

Theorem

The functors F and L induce an equivalence between the category $\overline{\text{QGPD}}$ and the full subcategory of WHQ of all pointed cosemisimple weak Hopf quasigroups.

Corollary

If $\mathbb{K}$ is algebraically closed, the functors F and L induce an equivalence between the category $\overline{\text{QGPD}}$ and the full subcategory of WHQ of all cocommutative cosemisimple weak Hopf quasigroups.

- In the associative setting (replace quasigroupoids by small groupoids with finitely many objects), the previous results are the ones proved in

G. Böhm, J. Gómez-Torrecillas, E. López Centella, On the category of weak bialgebras, **J. Algebra** 399 (2014), 801-844.

for weak Hopf algebras.

Corollary

The category of I.P. loops is equivalent to the one of pointed cosemisimple Hopf quasigroups.

Corollary

If $\mathbb{K}$ is algebraically closed, the category of I.P. loops is equivalent to the one of cocommutative cosemisimple Hopf quasigroups.

Corollary

- (i) The category of groups is equivalent to the one of pointed cosemisimple Hopf algebras.
- (ii) If $\mathbb{K}$ is algebraically closed, the category of groups is equivalent to the one of cocommutative cosemisimple Hopf algebras.

Matched pairs of quasigroupoids

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Definition

Let A and H be quasigroupoids with products $\bullet$ and $\star$ and with the same base. A **left action** of H on A is a map $\varphi_A : H_1 \times_{s_H} t_A A_1 \rightarrow A_1$, satisfying:

- (1) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$, $t_A(\varphi_A(h, a)) = t_H(h)$.
- (2) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$, $(g, h) \in H_1 \times_{s_H} t_H H_1$, $\varphi_A(g \star h, a) = \varphi_A(g, \varphi_A(h, a))$.
- (3) For all $a \in A_1$, $\varphi_A(id_H(t_A(a)), a) = a$.

Similarly, a **right action** of A on H is a map $\phi_H : H_1 \times_{s_H} t_A A_1 \rightarrow H_1$, satisfying:

- (1) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$, $s_H(\phi_H(h, a)) = s_A(a)$.
- (2) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$, $(a, b) \in A_1 \times_{s_A} t_A A_1$, $\phi_H(h, a \bullet b) = \phi_H(\phi_H(h, a), b)$,
- (3) For all $h \in H_1$, $\phi_H(h, id_A(s_H(h))) = h$.

Definition

A **matched pair of quasigroupoids** is a pair of quasigroupoids (A, H) with the same base together with a left action of H on A , $\varphi_A : H_1 \times_{s_H} t_A A_1 \rightarrow A_1$, and a right action of A on H , $\phi_H : H_1 \times_{s_H} t_A A_1 \rightarrow H_1$, satisfying the following properties:

(1) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$,

$$s_A(\varphi_A(h, a)) = t_H(\phi_H(h, a)).$$

(2) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$ and $(a, b) \in A_1 \times_{s_A} t_A A_1$,

$$\varphi_A(h, a \bullet b) = \varphi_A(h, a) \bullet \varphi_A(\phi_H(h, a), b).$$

(3) For all $(h, a) \in H_1 \times_{s_H} t_A A_1$ and $(g, h) \in H_1 \times_{s_H} t_H H_1$,

$$\phi_H(g \star h, a) = \phi_H(g, \varphi_A(h, a)) \star \phi_H(h, a).$$

Theorem

Let (A, H) be a matched pair of quasigroupoids. For all $a, b \in A_1$, $g, h \in H_1$ for which the operations are defined, the following identities hold:

$$\varphi_A(h, id_A(s_H(h))) = id_A(t_H(h)),$$

$$\phi_H(id_H(t_A(a)), a) = id_H(s_A(a)),$$

$$\lambda_A(\varphi_A(h, a)) = \varphi_A(\phi_H(h, a), \lambda_A(a)),$$

$$\lambda_H(\phi_H(h, a)) = \phi_H(\lambda_H(h), \varphi_A(h, a))$$

$$(b \bullet \varphi_A(h, a)) \bullet \varphi_A(\phi_H(h, a), \lambda_A(a)) = b,$$

$$\phi_H(\lambda_H(h), \varphi_A(h, a)) \star (\phi_H(h, a) \star g) = g,$$

$$\varphi_A(\lambda_H(\phi_H(h, a)), \lambda_A(\varphi_A(h, a))) = \lambda_A(a),$$

$$\phi_H(\lambda_H(\phi_H(h, a)), \lambda_A(\varphi_A(h, a))) = \lambda_H(h),$$

$$\lambda_A(a) \bullet \varphi_A(\lambda_H(h), b) = \varphi_A(\lambda_H(\phi_H(h, a)), \lambda_A(\varphi_A(h, a)) \bullet b),$$

$$\phi_H(g, \lambda_A(a)) \star \lambda_H(h) = \phi_H(g \star \lambda_H(\phi_H(h, a)), \lambda_A(\varphi_A(h, a))).$$

Theorem

Let (A, H) be a matched pair of quasigroupoids. The pair

$$A \bowtie H = ((A \bowtie H)_0, (A \bowtie H)_1),$$

where $(A \bowtie H)_0 = A_0$, $(A \bowtie H)_1 = A_1 \times_{s_A} {}_{t_H} H_1$, is a quasigroupoid with source morphism $s_{A \bowtie H}(a, h) = s_H(h)$, target morphism $t_{A \bowtie H}(a, h) = t_A(a)$, identity map $id_{A \bowtie H}(x) = (id_A(x), id_H(x))$, product

$$(a, g) \cdot_{\Psi} (b, h) = (a \bullet \Psi_1(g, b), \Psi_2(g, b) \star h),$$

where $\Psi : H_1 \times_{s_H} {}_{t_A} A_1 \rightarrow A_1 \times_{s_A} {}_{t_H} H_1$ is the map with components $\Psi_1(g, b) = \varphi_A(g, b)$ and $\Psi_2(g, b) = \phi_H(g, b)$, and inverse map

$$\lambda_{A \bowtie H}(a, h) = \Psi(\lambda_H(h), \lambda_A(a)).$$

Definition

Let (A, H) be a matched pair of quasigroupoids. The quasigroupoid $A \bowtie H$ will be called the **double crossed product** or the **diagonal quasigroupoid** of A and H .

Theorem

Let (A, H) be a matched pair of quasigroupoids. Then, A and H are subquasigroupoids of $A \bowtie H$ where

$$i^A : A \rightarrow A \bowtie H$$

is defined by $i^A = (i_0^A : A_0 \rightarrow (A \bowtie H)_0, i_1^A : A_1 \rightarrow (A \bowtie H)_1)$ with

$$i_0^A = id_{A_0}, \quad i_1^A(a) = (a, id_H(s_A(a))),$$

and $i^H : H \rightarrow A \bowtie H$ is defined by $i^H = (i_0^H : H_0 \rightarrow (A \bowtie H)_0, i_1^H : H_1 \rightarrow (A \bowtie H)_1)$ where

$$i_0^H = id_{A_0}, \quad i_1^H(g) = (id_A(t_H(g)), g),$$

Example

Let A be a I.P. loop with product $\cdot$ and let X be a set. Assume that there exists an action of A over X denoted by $\psi_X : A \times X \rightarrow X$. Let $B = (B_0, B_1)$ be the quasigroupoid associated to the action ψ_X . Let X^d be the discrete groupoid. Then, (X^d, B) is a matched pair of quasigroupoids where

$$\varphi_{X^d}((a, x), x) = \psi_X(a, x), \quad \phi_B((a, x), x) = (a, x).$$

In this case the double crossed product $X^d \bowtie B$, is defined by

$$(X^d \bowtie B)_0 = X, \quad (X^d \bowtie B)_1 = X_1^d \, s_{X^d} \times_{t_B} B_1,$$

$$s_{X^d \bowtie B}(y, (a, x)) = x, \quad t_{X^d \bowtie B}(y, (a, x)) = y, \quad id_{X^d \bowtie B}(x) = (x, (e_A, x)),$$

$$(y, (a, x)) \cdot_{\psi} (x, (b, t)) = (y, (a \cdot b, t))$$

and

$$\lambda_{X^d \bowtie B}(y, (a, x)) = (\psi_X(a^{-1}, y), (a^{-1}, y)).$$

Definition

Let A , H and B be quasigroupoids with the same base. Assume that A and H are subquasigroupoids of B with associated inclusions $i^A : A \rightarrow B$ and $i^H : H \rightarrow B$ satisfying that $i_0^A = i_0^H = id_{A_0}$. Let $\diamond$ be the product defined in B . Then we will say that $[A, H]$ is an **exact factorization of B** , if in the case that the products involved are defined, we have that

- (1) $i^H(g) \diamond (i^A(a) \diamond i^A(b)) = (i^H(g) \diamond i^A(a)) \diamond i^A(b)$,
- (2) $i^H(g) \diamond (i^H(h) \diamond i^A(a)) = (i^H(g) \diamond i^H(h)) \diamond i^A(a)$,
- (3) $i^H(h) \diamond (i^A(a) \diamond i^H(f)) = (i^H(h) \diamond i^A(a)) \diamond i^H(f)$,
- (4) $i^A(c) \diamond (i^H(h) \diamond i^A(a)) = (i^A(c) \diamond i^H(h)) \diamond i^A(a)$,
- (5) $i^A(a) \diamond (i^A(b) \diamond i^H(g)) = (i^A(a) \diamond i^A(b)) \diamond i^H(g)$,
- (6) $i^A(a) \diamond (i^H(g) \diamond i^H(h)) = (i^A(a) \diamond i^H(h)) \diamond i^H(g)$

hold and

- (7) The map $\theta_B = \diamond \circ (i^A \times i^H) : A_1 \times_{s_A \times t_H} H_1 \rightarrow B_1$ is a bijection.

Theorem

Let $[A, H]$ be an exact factorization of a quasigroupoid B . Then, there exists a matched pair of quasigroupoids (A, H) and an isomorphism of quasigroupoids between $A \bowtie H$ and B .

Let (A, H) be a matched pair of finite quasigroupoids. Let φ_A be the left action of H on A and let ϕ_H be a right action of A on H . Then, we can define the following morphisms in $\mathbb{K}\mathbf{Vect}$ by:

$$\varphi_{\mathbb{K}[A]} : \mathbb{K}[H] \otimes \mathbb{K}[A] \rightarrow \mathbb{K}[A], \quad \varphi_{\mathbb{K}[A]}(h \otimes a) = \begin{cases} \varphi_A(h, a) & \text{if } (h, a) \in H_1 \times_{s_H} A_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_{\mathbb{K}[H]} : \mathbb{K}[H] \otimes \mathbb{K}[A] \rightarrow \mathbb{K}[A], \quad \phi_{\mathbb{K}[H]}(h \otimes a) = \begin{cases} \phi_H(h, a) & \text{if } (h, a) \in H_1 \times_{s_H} A_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Phi : \mathbb{K}[H] \otimes \mathbb{K}[A] \rightarrow \mathbb{K}[A] \otimes \mathbb{K}[H]$ be the morphism defined by

$$\Phi = (\varphi_{\mathbb{K}[A]} \otimes \phi_{\mathbb{K}[H]}) \circ (id_{\mathbb{K}[A]} \otimes c_{\mathbb{K}[H], \mathbb{K}[A]} \otimes id_{\mathbb{K}[H]}) \circ (\delta_{\mathbb{K}[H]} \otimes \delta_{\mathbb{K}[A]})$$

and let $\nabla_{\Phi} : \mathbb{K}[A] \otimes \mathbb{K}[H] \rightarrow \mathbb{K}[A] \otimes \mathbb{K}[H]$ where

$$\nabla_{\Phi} = (\mu_{\mathbb{K}[A]} \otimes id_{\mathbb{K}[H]}) \circ (id_{\mathbb{K}[A]} \otimes (\Phi \circ (id_{\mathbb{K}[H]} \otimes 1_{\mathbb{K}[A]}))).$$

Then,

$$\Phi(h \otimes a) = \begin{cases} \varphi_A(h, a) \otimes \phi_H(h, a) & \text{if } (h, a) \in H_1 \times_{s_H} A_1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nabla_\Phi(a \otimes h) = \begin{cases} a \otimes h & \text{if } (a, h) \in A_1 \times_{s_A} H_1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, ∇_Φ is an idempotent morphism with image

$$Im(\nabla_\Phi) = \langle \{a \otimes h \mid (a, h) \in A_1 \times_{s_A t_H} H_1\} \rangle.$$

If we denote by $\mathbb{K}[A] \bowtie \mathbb{K}[H]$ the image of ∇_Φ , in this $\mathbb{K}$ -vector space we can define a product by

$$\mu_{\mathbb{K}[A] \bowtie \mathbb{K}[H]} = (\mu_{\mathbb{K}[A]} \otimes \mu_{\mathbb{K}[H]}) \circ (id_{\mathbb{K}[A]} \otimes \Phi \otimes id_{\mathbb{K}[H]}).$$

Then,

$$\mu_{\mathbb{K}[A] \bowtie \mathbb{K}[H]}(a \otimes h \otimes b \otimes g) = \begin{cases} a \bullet \varphi_A(h, b) \otimes \phi_A(h, b) \star g & \text{if } (h, b) \in H_1 \times_{s_H t_A} A_1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Let (A, H) be a matched pair of finite quasigroupoids. Then

$$\mathbb{K}[A] \bowtie \mathbb{K}[H]$$

is a cocommutative weak Hopf quasigroup with unit

$$1_{\mathbb{K}[A] \bowtie \mathbb{K}[H]} = \sum_{x \in A_0} id_A(x) \otimes id_H(x),$$

product $\mu_{\mathbb{K}[A] \bowtie \mathbb{K}[H]}$ and where for all not null element $a \otimes h \in \mathbb{K}[A] \bowtie \mathbb{K}[H]$, the counit is defined by $\epsilon_{\mathbb{K}[A] \bowtie \mathbb{K}[H]}(a \otimes h) = 1$, the coproduct by

$$\delta_{\mathbb{K}[A] \bowtie \mathbb{K}[H]}(a \otimes h) = a \otimes h \otimes a \otimes h$$

and the antipode by

$$\lambda_{\mathbb{K}[A] \bowtie \mathbb{K}[H]}(a \otimes h) = \varphi_A(\lambda_H(h), \lambda_A(a)) \otimes \phi_H(\lambda_H(h), \lambda_A(a)).$$

Theorem

Let (A, H) be a matched pair of finite quasigroupoids. The cocommutative weak Hopf quasigroups $\mathbb{K}[A \bowtie H]$ and $\mathbb{K}[A] \bowtie \mathbb{K}[H]$ are isomorphic in WHQ.

Corollary

Let (A, H) be a matched pair of I.P. loops. The cocommutative Hopf quasigroups $\mathbb{K}[A \bowtie H]$ and $\mathbb{K}[A] \bowtie \mathbb{K}[H]$ are isomorphic in the category of Hopf quasigroups.



Thank you very much for your attention